The Price of Anarchy in Auctions
Part II: The Smoothness Framework

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Part II: High-level goals

- PoA in auctions (as games of incomplete information):
  - Single-Item First Price, All-Pay, Second Price Auctions
  - Simultaneous Single Item Auctions
  - Position Auctions: GSP, GFP
  - Combinatorial auctions
General Approach

- Reduce analysis of complex setting to simple setting.

- Conclusion for simple setting $X$, proved under restriction $P$, extends to complex setting $Y$
  
  - $X$: complete information PNE to $Y$: incomplete information BNE
  
  - $X$: single auction to $Y$: composition of auctions
Best-Response Analysis

- Objective in X is good because each player doesn’t want to deviate to strategy $b_i'$

- Extension from setting X to setting Y: if best response argument satisfies condition P then conclusion extends to Y
First Extension Theorem

Complete info PNE to BNE with correlated values
- **Target setting.** First Price Bayes-Nash Equilibrium with asymmetric correlated values

- **Simple setting.** Complete information Pure Nash Equilibrium

- **Thm.** If proof of PNE PoA based on own-value based deviation argument then PoA of BNE also good

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**First Extension Theorem**

Complete info PNE to BNE with correlated values

References:

Roughgarden STOC’09
Lucier, Paes Leme EC’11
Roughgarden EC’12
Syrgkanis ‘12
Syrgkanis, Tardos STOC’13
First-Price Auction Refresher

- Highest bidder wins:
  \[ x_i(b) = \{\text{indicator that } i \text{ wins}\} \]
- Pays his bid: \( P_i(b) = b_i \cdot x_i(b) \)
- Quasi-Linear preferences:
  \[ \text{UTILITY} = \text{VALUE} - \text{PAYMENT} \]
  \[ u_i(b) = (v_i - b_i) \cdot x_i(b) \]
- Objective:
  \[ \text{WELFARE} = \text{UTILITIES} + \text{PAYMENTS} \]
  \[ SW(b) = \sum_i u_i(b) + \sum_i P_i(b) \]
  \[ = \sum_i (u_i(b) + b_i \cdot x_i(b)) = \sum_i v_i \cdot x_i(b) \]
First-Price Auction
Target: BNE with correlated values

- \( \mathbf{v} = (v_1, ..., v_n) \sim F \): correlated distribution

- Conditional on value, maximizes utility:
  \[
  E[u_i(b(v))|v_i] \geq E[u_i(b'_i, b_{-i}(v_{-i}))|v_i]
  \]

- Equilibrium Welfare:
  \[
  E[SW(b(v))] = E \left[ \sum_i v_i \cdot x_i(b(v)) \right]
  \]

- Optimal Welfare: highest value bidder
  \[
  E[OPT(v)] = E \left[ \sum_i v_i \cdot x^*_i(v) \right]
  \]
First-Price Auction
Target: BNE with correlated values

\[ PoA = \frac{E[OPT(v)]}{E[SW(b(v))]} \]
First-Price Auction
Simpler: PNE and complete Information

- $v = (v_1, ..., v_n)$: common knowledge
- $b_i$ maximizes utility:
  $$u_i(b) \geq u_i(b'_i, b_{-i})$$
- Equilibrium Welfare:
  $$SW(b) = \sum_i v_i \cdot x_i(b)$$
- Optimal Welfare:
  $$OPT(v) = \sum_i v_i \cdot x_i^*(v)$$
First-Price Auction
Simpler: PNE and complete Information

\[ P_{oA} = \frac{OPT(\mathbf{v})}{SW(\mathbf{b})} \]

\[ p(\mathbf{b}) = \max_i b_i \]
First-Price Auction
Simpler: PNE and complete Information

**Theorem.** PoA = 1

**Proof.** Highest value player can deviate to $p(b)^+$

$$u_1(p(b)^+, b_{-i}) = v_1 - p(b)^+$$

$$u_i(0, b_{-i}) = 0$$

$$\sum_i u_i(b) \geq \sum_i u_i(b'_i, b_{-i}) = v_1 - p(b)$$

By PNE condition
**Theorem.** $PoA = 1$

**Proof.** Highest value player can deviate to $p(b)^+$

$$u_1(p(b)^+, b_{-i}) = v_1 - p(b)^+$$

$$u_i(0, b_{-i}) = 0$$

$$\text{UTIL}(b) \geq \sum_i u_i(b'_i, b_{-i}) = v_1 - \text{REV}(b)$$

$$\text{UTIL}(b) + \text{REV}(b) \geq v_1$$

$$SW(b) \geq v_1$$
Direct extensions

- What if conclusions for PNE of complete information directly extended to:
  - incomplete information BNE
  - simultaneous composition of single-item auctions

- Obviously: $PoA = 1$ doesn’t carry over

- Possible, but we need to restrict the type of analysis
Problem in previous PNE proof

- **Recall.** PoA = 1 because highest value player doesn’t want to deviate to $p^+$
- **Challenge.** Don’t know $p$ or $v_{-i}$ in incomplete information
- **Idea.** Restrict deviation to not depend on these parameters
**First-Price Auction**

**Simpler: PNE and complete Information**

Recall PoA=1 Proof

**Proof.** Highest value player can deviate to $p(b)^+$

$$u_1(p(b)^+, b_{-i}) = v_1 - p(b)^+$$

$$u_i(0, b_{-i}) = 0$$

$$U(b) = \sum_i u_i(b'_i, b_{-i}) = v_1 - \text{REV}(b)$$

$$U(b) + \text{REV}(b) \geq v_1$$

$$SW(b) \geq v_1$$

Can we find $b'_i$ that depend only on $v_i$?
Own-value deviations
(price and other values oblivious)

New Theorem. $PoA \leq 2$

Proof. Each player can deviate to $b'_i = \frac{v_i}{2}$
Own-value deviations
(price and other values oblivious)

New Theorem. PoA \leq 2

Proof. Each player can deviate to \( b_i' = \frac{v_i}{2} \)
Own-value deviations
(price and other values oblivious)

New Theorem. \( P\text{o}A \leq 2 \)

Proof. Each player can deviate to \( b'_i = \frac{v_i}{2} \)

\[
\begin{align*}
    u_i \left( \frac{v_i}{2}, b_{-i} \right) + p(b) &\geq \frac{v_i}{2} \\
    p(b) &= \max b_i
\end{align*}
\]
Own-value deviations
(price and other values oblivious)

New Theorem. PoA ≤ 2

Proof. Each player can deviate to $b'_i = \frac{v_i}{2}$

$$u_i \left(\frac{v_i}{2}, b_{-i}\right) + p(b) \cdot x^*_i(v) \geq \frac{v_i}{2} \cdot x^*_i(v)$$

$$\text{UTIL}(b) \geq \sum_i u_i \left(\frac{v_i}{2}, b_{-i}\right) + p(b) \geq \frac{1}{2} \text{OPT}(v)$$

$$\text{UTIL}(b) + \text{REV}(b) \geq \frac{1}{2} \text{OPT}(v)$$

$$SW(b) \geq \frac{1}{2} \text{OPT}(v)$$
New Theorem.

Proof. Each player can deviate to $b'_i = v_i \cdot 2u_i v_i^2$, $b_i + p b \cdot x^*_i v \geq v_i^2 \cdot x^*_i (v_{i-1} v_i^2)$, $p(b) = \max_i b_i$, $UTIL(b) \geq \sum_i u_i(b'_i, b_{-i}) + REV(b) \geq \frac{1}{2} OPT(v)$

$UTIL(b) + REV(b) \geq \frac{1}{2} OPT(v)$  
$SW(b) \geq \frac{1}{2} OPT(v)$
\((\lambda, \mu)\) – Smoothness via own-value deviations

Exists \(b'_i\) depending only on own value

For any bid vector \(b\)

\[
\sum_{i} u_i(b'_i, b_{-i}) + \mu \cdot \text{REV}(b) \geq \lambda \cdot \text{OPT}(v)
\]
$(\lambda, \mu)$ – Smoothness via own-value deviations

Exists $b'_i$ depending only on own value

For any bid vector $\mathbf{b}$

$$\sum_i u_i(b'_i, b_{-i}) + \mu \cdot REV(\mathbf{b}) \geq \lambda \cdot OPT(\mathbf{v})$$

Note. Smoothness is property of auction not equilibrium
\((\lambda, \mu)\) – Smoothness via own-value deviations

Exists \(b_i'\) depending only on own value

For any bid vector \(\mathbf{b}\)

\[
\sum_i u_i(b_i', b_{-i}) + \mu \cdot \text{REV}(\mathbf{b}) \geq \lambda \cdot \text{OPT}(\mathbf{v})
\]

Applies to any auction: Not First-Price Auction specific
$(\lambda, \mu)$ – Smoothness implies $\text{PoA} \leq \mu / \lambda$

**Proof.** If $b$ PNE then

$$\text{UTIL}(b) + \mu \cdot \text{REV}(b) \geq \sum_{i} u_i(b_i', b_{-i}) + \mu \cdot \text{REV}(b) \geq \lambda \cdot \text{OPT}(v)$$

**Note.** $\text{UTIL}(b) = \text{SW}(b) - \text{REV}(b)$

**Note.** $\text{SW}(b) \geq \text{REV}(b)$

$$\text{SW}(b) + (\mu - 1) \cdot \text{REV}(b) \geq \lambda \cdot \text{OPT}(v)$$

$$\text{SW}(b) + (\mu - 1) \cdot \text{SW}(b) \geq \lambda \cdot \text{OPT}(v)$$

$$\mu \cdot \text{SW}(b) \geq \lambda \cdot \text{OPT}(v)$$
Finally

**First Extension Theorem.** If PNE PoA proved by showing \((\lambda, \mu)\) – smoothness property via own-value deviations, then PoA bound extends to BNE with correlated values.

*Note. Not specific to First-Price Auction*
$(\lambda, \mu)$ – Smoothness implies BNE PoA $\leq \frac{\mu}{\lambda}$

**Proof.** If $b(\cdot)$ BNE then $E[u_i(b(v))] \geq E\left[u_i\left(\frac{v_i}{2}, b_{-i}(v_{-i})\right)\right]$

$E_v\left[\text{UTIL}(b) + \mu \cdot \text{REV}(b) \geq \sum_i u_i(b'_i, b_{-i}) + \mu \cdot \text{REV}(b) \geq \lambda \cdot \text{OPT}(v)\right]$  

Just redo PNE proof in expectation over values.
Optimizing over \((\lambda, \mu)\)

- Is half value best own-value deviation?
- Bid \(b_i' \sim H(v_i)\) with support \(\left[0, \left(1 - \frac{1}{e}\right)v_i\right]\) and
  \[
  h(b_i') = \frac{1}{v_i - b_i'}
  \]

\[
p(b) = \max_i b_i
\]
Optimizing over $(\lambda, \mu)$

- Bid $b'_i \sim H(v_i)$ with support $\left[0, \left(1 - \frac{1}{e}\right)v_i\right]$ and $h(b'_i) = \frac{1}{v_i - b'_i}$

$$x(b_i, b_{-i})$$

$$b'_1 \sim H(v_1)$$

$$b'_i \sim H(v_i)$$

$$b'_n \sim H(v_n)$$

$$p(b) = \max_i b_i$$
Optimizing over \((\lambda, \mu)\)

- Bid \(b_i' \sim H(v_i)\) with support \([0, (1 - \frac{1}{e})v_i]\) and \(h(b_i') = \frac{1}{v_i - b_i'}\)

\[
x(b_i, b_{-i})
\]

\[
p(b) = \max_i b_i
\]

\[
\left(1 - \frac{1}{e}\right)v_i
\]

w.p. \(\frac{1}{v_i - b_i'}\)
Optimizing over \((\lambda, \mu)\)

- Bid \(b'_i \sim H(v_i)\) with support \([0, (1 - \frac{1}{e})v_i]\) and \(h(b'_i) = \frac{1}{v_i-b'_i}\)

\[
\begin{align*}
x(b_i, b_{-i}) & = p(b) E[u_i(b'_i)] \\
p(b) & = \max_i b_i \\
E[u_i(b'_i)] & + p(b) > \left(1 - \frac{1}{e}\right)v_i
\end{align*}
\]

- So in fact: \((1 - \frac{1}{e}, 1)\)-smooth. \(PoA \leq \frac{e}{e-1} \approx 1.58\)
RECAP

- **First Extension Thm.** If proof of PNE PoA based on \((\lambda, \mu)\) — smoothness via own-value based deviations then PoA of BNE with correlated values also \(\mu/\lambda\)

QUESTIONS?
Second Extension Theorem

Single auction to simultaneous auctions

PNE complete information
- **Target setting.** Simultaneous single-item first price auctions with unit-demand bidders (complete information PNE).

- **Simple setting.** Single-item first price auction (complete information PNE).

- **Thm.** If proof of PNE PoA of single-item based on proving $(\lambda, \mu)$-smoothness via own-value deviation then PNE PoA of simultaneous auctions also $\mu/\lambda$.

References:
Roughgarden STOC’09
Roughgarden EC’12
Syrgkanis ‘12
Syrgkanis, Tardos STOC’13
Simultaneous First-Price Auctions
Unit-demand bidders

Unit-Demand Valuation
\[ v_i(S) = \max_{j \in S} v_i^j \]
Simultaneous First-Price Auctions
Unit-demand bidders

Unit-Demand Valuation
\[ v_i(S) = \max_{j \in S} v_i^j \]
Simultaneous First-Price Auctions

Can we derive global efficiency guarantees from local $(\frac{1}{2}, 1)$—smoothness of each first price auction?

**APPROACH:** Prove smoothness of the global mechanism

**GOAL:** Construct global deviation

**IDEA:** Pick your item in the optimal allocation and perform the smoothness deviation for your local value $v_i^j$, i.e. $\frac{v_i^j}{2}$
Smoothness locally:

\[ u_i(b'_i, b_{-i}) + p_{j_i}(b) \geq \frac{v_{j_i}^i}{2} \]

Summing over players:

\[ \sum_i u_i(b'_i, b_{-i}) + REV(b) \geq \frac{1}{2} \cdot OPT(v) \]

Implying \((\frac{1}{2}, 1)\)–smoothness property globally.
Second Extension Theorem. If proof of PNE PoA of single-item auction based on proving \((\lambda, \mu)\)-smoothness smoothness via own-value deviation then PNE PoA of simultaneous auctions also \(\leq \mu/\lambda\).
BNE PoA?

- BNE PoA of simultaneous single-item auctions with correlated unit-demand values $\leq 1/2$?

- Not really: deviation not oblivious to opponent valuations

- Item in the optimal matching depends on values of opponents
But Half-way there

- What we showed:

Exists $b'_i$ depending only on valuation profile $v$ (not $b_{-i}$)

For any bid vector $b$

$$\sum_i u_i(b'_i, b_{-i}) + \mu \cdot REV(b) \geq \lambda \cdot OPT(v)$$
RECAP

**Second Extension Theorem.** If proof of PNE PoA of single-item auction based on proving $(\lambda, \mu)$-smoothness then PNE PoA of simultaneous auctions also $\leq \mu/\lambda$.

Next we will extend above to BNE

QUESTIONS?
Third Extension Theorem
Complete info PNE to BNE with independent values
- **Target setting.** First Price Bayes-Nash Equilibrium with asymmetric independent values

- **Simple setting.** Complete information Pure Nash Equilibrium

- **Thm.** If proof of PNE PoA based on \((\lambda, \mu)\)-smoothness via valuation profile dependent deviation then PoA of BNE with independent values also \(\mu/\lambda\)

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**Third Extension Theorem**

Complete info PNE to BNE with independent values

References:
Christodoulou et al. ICALP’08
Roughgarden EC’12
Syrgkanis ‘12
Syrgkanis, Tardos STOC’13
Does this extend to BNE PoA?

\((\lambda, \mu)\) – Smoothness via valuation profile deviations

Exists \(b'_i\) depending only on valuation profile \(v\) (not \(b_{-i}\))

For any bid vector \(b\)

\[
\sum_i u_i(b'_i, b_{-i}) + \mu \cdot \text{REV}(b) \geq \lambda \cdot \text{OPT}(v)
\]
Recall First Extension Theorem.

If PNE PoA proved by showing \((\lambda, \mu)\) – smoothness property via own-value deviations, then PoA bound extends to BNE with correlated values.

- Relax First Extension Theorem to allow for dependence on opponents values

- To counterbalance: assume independent values
**BNE (independent valuations)**

- Need to construct feasible BNE deviations
- Each player random samples the others values and deviates as if that was the true values of his opponents
- Above works out, due to independence of value distributions
BNE (independent valuations)

\[ E\left[u_i^{v_i}(b'_i(v_i, w_{-i}), b_{-i}(v_{-i}))\right] = E\left[u_i^{w_i}(b'_i(w), b_{-i}(v_{-i}))\right] \]

Utility of deviation of player \(i\)
In expectation over his own value too.

Utility of deviation from a random sample of player \(i\) who knows the values of all other players.

**But where players play non equilibrium strategies.**
BNE (independent valuations)

$$E[u_i^{v_i}(b'_i(v_i, w_{-i}), b_{-i}(v_{-i}))] = E[u_i^{w_i}(b'_i(w), b_{-i}(v_{-i}))]$$

Utility of deviation of player $i$
In expectation over his own value too.

Utility of deviation from a random sample of player $i$ who knows the values of all other players.

But where players play non equilibrium strategies.

$$F_i \sim w_i$$

$$b'_i(w) = \frac{1}{2} \cdot v^*_i(w)$$

$$w_{-i} \sim F_i$$
BNE (independent valuations)

\[
\sum_i E[u_i^{v_i}(b'_i(v_i, w_{-i}), b_{-i}(v_{-i}))] = E \left[ \sum_i u_i^{w_i}(b'_i(w), b_{-i}(v_{-i})) \right]
\]

Sum of deviating utilities

Sum of complete information setting deviating utilities

\[b'_i(w) = \frac{1}{2} \cdot v^*_i(w)\]

\[F_i \sim w_i\]

\[w_{-i} \sim F_i\]
Recall. Exists $b'_i$ depending only on valuation profile $v$ (not $b_{-i}$)

For any bid vector $b$

$$\sum_{i} u_i(b'_i, b_{-i}) + \mu \cdot REV(b) \geq \lambda \cdot OPT(v)$$

Utility of deviation of player $i$

$$\sum_{i} u_i(b'_i, b_{-i}) \geq E\left[ \sum_{i} u_i w_i (b'_i(w), b_{-i}(v_{-i})) \right] = E \left[ \sum_{i} u_i w_i (b'_i(w), b_{-i}(v_{-i})) \right] \geq E\left[ \lambda \cdot OPT(w) - \mu \cdot REV(b(v)) \right]$$

By smoothness on the left
BNE (independent valuations)

\[
\sum_i E[u_i^{v_i}(b'_i(v_i, w_{-i}), b_{-i}(v_{-i}))] = E \left[ \sum_i u_i^{w_i}(b'_i(w), b_{-i}(v_{-i})) \right] \\
\geq E[\lambda \cdot OPT(w) - \mu \cdot REV(b(v))]
\]

**Found** \(b'_i\) that depend only on \(v_i\) such that:

\[
\sum_i E[u_i(b'_i(v_i), b_{-i}(v_{-i}))] + \mu \cdot E[REV(b(v))] \geq \lambda \cdot E[OPT(v)]
\]

Rest is easy
**Third Extension Theorem.** If PNE PoA proved by showing \((\lambda, \mu)\) –smoothness property via valuation profile dependent deviations, then PoA bound extends to BNE with independent values.
RECAP

- **Thm.** If proof of PNE PoA based on \((\lambda, \mu)\)-smoothness via valuation profile dependent deviation then PoA of BNE with independent values also \(\mu/\lambda\)

- **Corollary.** If PNE PoA of single-item auction proved via \((\lambda, \mu)\)-smoothness via valuation profile dependent deviation, then BNE of simultaneous auctions with unit-demand and independent also \(\mu/\lambda\)

**Third Extension Theorem**

Complete info PNE to BNE with independent values
RECAP

- **Thm.** If proof of PNE PoA based on $(\lambda, \mu)$-smoothness via valuation profile dependent deviation then PoA of BNE with independent values also $\mu/\lambda$

- **Corollary.** If PNE PoA of single-item auction proved via $(\lambda, \mu)$-smoothness via valuation profile dependent deviation, then BNE of simultaneous auctions with *submodular* and independent also $\mu/\lambda$

- **Corollary.** BNE PoA of simultaneous first price auctions with submodular bidders $\leq \frac{e}{e-1}$

QUESTIONS?
Direct approach: Arguing about distributions
Focusing on complete info PNE, might be restrictive in some settings.

Working with the distributions directly can potentially yield better bounds.

Direct approach
Arguing about distributions

References:
Feldman et al. STOC’13
Price of the item follows a distribution D

What if a player deviates to bidding a random sample from price distribution

The probability that he wins is \( \frac{1}{2} \) by symmetry of the two distributions

He pays at most \( E[p] \)

\[
E[u_i(b'_i, b_{-i}(v_{-i}))] \geq \frac{v_i}{2} - E[p]
\]
- Same spirit: exists deviations that depend on price distribution such that
  \[
  \sum_i E[u_i(b'_i, b_{-i}(v_{-i}))] + E[REV(b(v))] \geq \frac{E[OPT(v)]}{2}
  \]
- BNE PoA \leq 2
What does it buy us

- Correlated deviating strategies across multiple auctions
- Decomposition of deviation analysis to separate deviations imposes independent randomness
- Correlation can achieve higher deviating utility

Sub-additive valuations
\[ v_i(S) + v_i(T) \geq v_i(S \cup T) \]
What does it buy us

- Correlation can achieve higher deviating utility

Sub-additive valuations
\[ v_i(S) + v_i(T) \geq v_i(S \cup T) \]

- Draw bid from price distribution

- \( X(b, p) \): set of won items with bid vector \( b \) and price vector \( p \)

- Either I win or price wins:
\[ X(b, p) + X(p, b) = S \]

- By symmetry:
\[ E[v(X(b', p))] = E[v(X(p, b'))] \]

- Value collected:
\[ E[v(X(b', p))] = \frac{1}{2} E[v(X(b', p)) + v(X(p, b'))] \geq \frac{1}{2} E[v(S)] \]
• Drawing deviation from price distribution!

• Buys correlation across auctions

• Better bounds beyond submodular
Second Price Payment Rules
Second price

Vickrey Auction - Truthful, efficient, simple (second price)

but has many bad Nash equilibria

Assume bid ≤ value (no overbidding)

Theorem. All Nash equilibria efficient. highest value wins
Second Price and Overbidding

- Same approach but replace Payments with “Winning Bids” and use no-overbidding

For any bid vector $\mathbf{b}$

$$\sum_i u_i(b'_i, b_{-i}) + \mu \cdot BIDS(\mathbf{b}) \geq \lambda \cdot OPT(\mathbf{v})$$

- No overbidding assumption:

$$BIDS \leq WELFARE$$

Then $PoA \leq \frac{1+\mu}{\lambda}$
Smoothness of Vickrey Auction

- Deviate to bidding your value: \( b'_i(v_i) = v_i \)

- \( B(b) \): winning bid

- Either winning bid \( B(b) \geq v_i \) or \( u_i(b'_i, b_{-i}) = v_i - B_i(b) \)

\[
u_i(b'_i, b_{-i}) + B_i(b) \geq v_i \implies u_i(b'_i, b_{-i}) + B_i(b) \cdot x^*_i(v) \geq v_i \cdot x^*_i(v)
\]

\[
\sum_i u_i(b'_i, b_{-i}) + BIDS(b) \geq OPT(v)
\]
Smoothness of Vickrey Auction

- Vickrey auction (1,1)-smooth using bids
- $PoA \leq 2$: under no-overbidding

- Vickrey is efficient?

- $PoA \leq 2$: extends to simultaneous Vickrey auctions even under BNE with independent values
Sneak Peek of Examples
Generalized First-Price Auction

- Allocate slots by bid
- Charge bid per-click
- Utility: \( u_i(b) = a_{\sigma(i)}(v_i - b_i) \)

Advertisers

\( v_1 \sim F_1 \)
\( v_i \sim F_i \)
\( v_n \sim F_n \)

Slots

\( b_1 \)
\( b_i \)
\( b_n \)

CTRs

\( a_1 \)
\( a_2 \)
\( a_3 \)
\( a_4 \)
Allocated items greedily to highest remaining bid

If allocated item $j(b)$, charge $b_i^{j(b)}$

Utility:
\[ u_i(b) = v_i^{j(b)} - b_i^{j(b)} \]
Single-Minded Bidders

1

2
Single-minded: $v_i$ for whole set $S_i$

3

Items

$S_1$

$S_2$

$S_3$

- Each bidder submits $b_i$ and $T_i$

- Run some algorithm (optimal or greedy $O(\sqrt{m})$-approx.) over reported single-minded values

- Charge bid $b_i$ if allocated
Examples

**GFP**
- Allocate slots by bid
- Charge bid per-click
- Utility: \( u_i(b) = a_{\sigma(i)}(v_i - b_i) \)

**Matching Markets-Greedy Allocation**
- Allocated items greedily to highest remaining bid
- If allocated item \( j(b) \), charge \( b_i^{j(b)} \)
- Utility: \( u_i(b) = v_i^{j(b)} - b_i^{j(b)} \)

**Single-Minded Combinatorial Auctions**
- Each bidder submits \( b_i \) and \( T_i \)
- Run some algorithm (optimal or greedy \( O(\sqrt{m}) \)-approx.) over reported single-minded values
- Charge bid \( b_i \) if allocated
Examples
Generalized First-Price Auction

- Allocate slots by bid
- Charge bid per-click
- Utility: \( u_i(b) = a_{\sigma(i)}(v_i - b_i) \)

Advertisers

- \( v_1 \sim F_1 \)
- \( v_i \sim F_i \)
- \( v_n \sim F_n \)

Slots

- \( a_1 \)
- \( a_2 \)
- \( a_3 \)

CTRs

- \( b_1 \)
- \( b_i \)
- \( b_n \)
Smoothness of GFP

\[ \sum_i u_i(b'_i, b_{-i}) + \mu \cdot \text{REV}(b) \geq \lambda \cdot \sum_i a_{\text{opt}(i)}v_i \]

- \( b'_i = \frac{v_i}{2} \)
- Either bid of player at slot \( \text{opt}(i) \) \( \geq \frac{v_i}{2} \)
- Or utility \( \geq \frac{a_{\text{opt}(i)}v_i}{2} \)

\[
\sum_i u_i \left( \frac{v_i}{2}, b_{-i} \right) + a_{\text{opt}(i)} \cdot b_{\pi(\text{opt}(i))} \geq \frac{a_{\text{opt}(i)}v_i}{2}
\]

\[
\sum_i u_i \left( \frac{v_i}{2}, b_{-i} \right) + \sum_i a_{\text{opt}(i)} \cdot b_{\pi(\text{opt}(i))} \geq \sum_i \frac{a_{\text{opt}(i)}v_i}{2}
\]

\[
\sum_i u_i \left( \frac{v_i}{2}, b_{-i} \right) + \text{REV}(b) \geq \frac{1}{2} \cdot \text{OPT}(v)
\]
Smoothness of GFP

\[
\sum_{i} u_i(b'_i, b_{-i}) + \mu \cdot REV(b) \geq \lambda \cdot \sum_{i} a_{opt(i)} v_i
\]

\[
\sum_{i} u_i \left( \frac{v_i}{2}, b_{-i} \right) + REV(b) \geq \frac{1}{2} \cdot OPT(v)
\]

**Thm.** PoA \leq 2

**Proof.**

\[
\sum_{i} u_i(b) \geq \sum_{i} u_i \left( \frac{v_i}{2}, b_{-i} \right)
\]

\[
UTIL(b) + REV(b) \geq \frac{1}{2} \cdot OPT(v)
\]

\[
SW(v) \geq \frac{1}{2} \cdot OPT(v)
\]
Smoothness of GFP

\[ \sum_{i} u_i(b'_i, b_{-i}) + \mu \cdot \text{REV}(b) \geq \lambda \cdot \sum_{i} a_{opt(i)} v_i \]

Thm. Bayes-Nash PoA \leq 2

Proof.

\[ \sum_{i} E[u_i(b(v))] \geq \sum_{i} E \left[u_i \left(\frac{v_i}{2}, b_{-i}(v_{-i})\right)\right] \]

\[ E[\text{UTIL}(b(v))] + E[\text{REV}(b(v))] \geq \frac{1}{2} \cdot E[\text{OPT}(v)] \]

\[ E[\text{SW}(b(v))] \geq \frac{1}{2} \cdot E[\text{OPT}(v)] \]
Matching Markets – Greedy Mechanism

- Allocated items greedily to highest remaining bid

- If allocated item \( j(b) \), charge \( b_i^{j(b)} \)

- Utility:
  \[
  u_i(b) = v_i^{j(b)} - b_i^{j(b)}
  \]
Matching Markets – Greedy Mechanism

Unit-Demand Bidders

Items

- Deviation

\[ b_i^j = \frac{v_i^j}{2} \]

- Only for \( j \) = item in optimal matching

- If \( p_j(b) \) is price of item \( j \)

\[ u_i(b_i', b_{-i}) \geq \frac{v_i^j}{2} - p_j(b) \]

- Thus \((\frac{1}{2}, 1)\)-smooth via valuation profile dependent deviations

Unit-Demand

\[ v_i(S) = \max_{j \in S} v_i^j \]

Bidders

Items

80
Matching Markets – Greedy Mechanism

- In fact
  \[ b_i^j \sim H(v_i^j) \]
- Only for \( j \) = item in optimal matching
  \[ u_i(b_i', b_{-i}) \geq \left(1 - \frac{1}{e}\right)v_i^j - p_j(b) \]
- Thus \((1 - \frac{1}{e}, 1)\)-smooth
- Greedy on true values: 2-approx.
- Greedy on reported values: 1.58-approx.!
Incentives improve algorithmic approximation

- Greedy on true values: 2-approx.

At equilibrium:
- Player 2 never goes for first item
- Too expensive
- So allocation is efficient
Each bidder submits $b_i$ and $T_i$

Run some algorithm over reported single-minded values

Charge bid $b_i$ if allocated
Each bidder submits $b_i$ and $T_i$

Run **optimal algorithm** over reported single-minded values

Charge bid $b_i$ if allocated
Linear inefficiency!

$m$ Items

$v_1 = 1$

$v_2 = 1$

Single-minded: $v_i$ for whole set $S_i$

\[ S_1 = S_2 \]

At equilibrium:
- 1 and 2 bid $b = 1$, $T = [m]$
- Other players bid 0
- $SW = 1$ but $OPT = m$
Each bidder submits $b_i$ and $T_i$

Run $\sqrt{m}$ –Approximation Algorithm over reported single-minded values

Charge bid $b_i$ if allocated
\[ \sqrt{m} - \text{Approximation Algorithm} \]

Single-Minded Bidders

1

2

Single-minded: \( v_i \) for whole set \( S_i \)

3

Items

\[ \sqrt{m} - \text{Approximation Algorithm} \]

- Reweight bids as: \( \hat{b}_i = \frac{b_i}{\sqrt{|T_i|}} \)
- Allocate in decreasing order of \( \hat{b}_i \)
- Charge bid \( b_i \) if allocated

- Idea: A player can block at most \( \sqrt{m} \) other players of same value from being allocated
Bad Example Corrected

$m$ Items

$v_1 = 1$

$v_2 = 1$

Single-minded: $v_i$ for whole set $S_i$

$S_1 = S_2$

Large players cannot block all small players

$v = 1 - \epsilon$

$v = 1 - \epsilon$

$v = 1 - \epsilon$
Deviation $b'_i$: bid $\frac{v_i}{2}$ for $S_i$

Let $\tau_i(b)$: Threshold bid for being allocated $S_i$ (including bid of player)

By similar analysis:
$$u_i(b'_i, b_{-i}) + \tau_i(b) \geq \frac{v_i}{2}$$

Need to show: $\sum_i \tau_i(b) \leq c \cdot \text{REV}$
Smoothness of Approximation Algorithm

- Fact: Algorithm is $\sqrt{m}$-approximation

- Think of hypothetical situation where each bidder is duplicated
  - Duplicate bidder bids: $b_i = \tau_i(b) - \epsilon$ for set $S_i$

- By definition of $\tau_i(b)$: algorithm doesn’t allocate to them
  - Allocating to duplicate bidders yields welfare
    $$\sum_i \tau_i(b)$$

- Since algorithm is $\sqrt{m}$-approximation: $REV = \sum_i b_i X_i(b) \geq \frac{1}{\sqrt{m}} \sum_i \tau_i(b)$
Approximation improves efficiency

- Approximate mechanism: $\left(\frac{1}{2}, \sqrt{m}\right)$ – smooth

- Welfare at equilibrium $O(\sqrt{m})$-approximate NOT $O(m)$ – approximate
Some References

- **Smoothness**
  Roughgarden STOC’09, Lucier, Paes Leme EC’11, Roughgarden EC’12, Syrgkanis ‘12, Syrgkanis, Tardos STOC’13

- **Simultaneous First-Second Price Single-Item Auctions**
  Bikhchandani GEB’96, Christodoulou, Kovacs, Schapira ICALP’08, Bhawalkar, Roughgarden SODA’11, Hassidim, Kaplan, Mansour, Nisan EC’11, Feldman, Fu, Gravin, Lucier STOC’13

- **Auctions based on Greedy Allocation Algorithms**
  Lucier, Borodin SODA’10

- **AdAuctions (GSP, GFP)**
  Paes-Leme Tardos FOCS’10, Lucier, Paes-Leme + CKKK EC’11

- **Sequential First/Second Price Auctions**
  Paes Leme, Syrgkanis, Tardos SODA’12, Syrgkanis, Tardos EC’12

- **Multi-Unit Auctions**
  Bart de Keijzer et al. ESA’13

All above can be thought as smoothness proofs and some are compositions of auctions.
This conference

**Price of Anarchy in Auctions and Mechanisms**

- Dutting, Henzinger, Stanberger. Valuation Compressions in VCG-Based Combinatorial Auctions
- Jose R. Correa, Andreas S. Schulz and Nicolas E. Stier-Moses. The Price of Anarchy of the Proportional Allocation Mechanism Revisited
- Jason Hartline, Darrell Hoy and Sam Taggart. Interim Smoothness for Auction Welfare and Revenue. (poster)
- Michal Feldman, Vasilis Syrgkanis and Brendan Lucier. Limits of Efficiency in Sequential Auctions
- Brendan Lucier, Yaron Singer, Vasilis Syrgkanis and Eva Tardos. Equilibrium in Combinatorial Public Projects

**Price of Anarchy in Games**

- Xinran He and David Kempe. Price of Anarchy for the N-player Competitive Cascade Game with Submodular Activation Functions
- Mona Rahn and Guido Schäfer. Bounding the Inefficiency of Altruism Through Social Contribution Games
- Yoram Bachrach, Vasilis Syrgkanis and Milan Vojnovic. Incentives and Efficiency in Uncertain Collaborative Environments