# 1 No-Regret Learning in General Games and the Price of Anarchy

In the lecture 6, we learned about Correlated Equilibria which are more general solution concepts than Nash Equilibria. Furthermore, we saw the No-Swap Regret learning algorithms which converge to Correlated Equilibria, which are the extended versions of No-Regret learning algorithms for Nash Equilibria. In this lecture, we will learn concepts related to social welfare, price of anarchy, and no-regret learning in general games.

## 1.1 Utility Maximization Games and the Price of Anarchy

### 1.1.1 Social Welfare

So far we have been analyzing convergence to a solution concept. But, how good are those solution concepts in terms of quality, i.e. total user satisfaction?

Consider a (finite) game with $n$ players, where player $i$'s set of possible strategies is given by $\Sigma_i$. A strategy profile $s = (s_1, s_2, \cdots, s_n)$ is a vector of strategies such that $s_i \in \Sigma_i$. Let $\Sigma \coloneqq \Sigma_1 \times \cdots \times \Sigma_n$. The utility of player $i$ is denoted by $u_i(s)$ where $u_i : \Sigma \to \mathbb{R}$. We will measure the quality through the total utility of players, known as social welfare.

**Definition 1.** *The social welfare for a strategy profile $s \in \Sigma \coloneqq \Sigma_1 \times \cdots \times \Sigma_n$ is defined as the total utility of all players, i.e.*

$$SW(s) = \sum_i u_i(s).$$

We ask the following questions:

- How does the Nash Equilibrium social welfare compare to the optimal social welfare?
- What is the social welfare if players use strategies based on no-regret learning algorithms rather instead of playing Nash Equilibrium?

Consider an example of $n$ firms and $m$ markets. Each market $j$ has a total demand of $v_j$. Each firm's allowed strategy is to pick one market. If $n_j$ firms pick market $j$ then each firm gets an equal share $= \frac{v_j}{n_j}$. So if $n_j(s) = |\{i : s_i = j\}|$ then $u_i(s) = \frac{v_{s_i}}{n_{s_i}(s)}$. In this case, the social welfare of the $n$ firms is given by:

$$SW(s) = \sum_{j=1}^{m} \mathbb{1}\{\exists\ i \text{ that picks } j\}$$

Now, if all firms pick the high value market then each player gets utility 1. No one wants to deviate as they would get utility $1 - \epsilon$. Thus, social welfare when all the players play the Nash Equilibrium strategy, the social welfare is 3, i.e. SW(NE) = 3. However, the optimal social welfare is each to choose a separate market. In this case, SW(OPT) = 5 - $2\epsilon$. More generally, we will be interested in understanding how much more can the SW(OPT) be relative to the SW(NE).

### 1.1.2 Price of Anarchy

To characterize the inefficiency of a Nash Equilibrium (NE), the social welfare in the NE is compared with the optimal social welfare.

**Definition 2** (Koutsoupias - Papadimitriou)**.** *The Price of Anarchy (PoA) of a game is defined as the maximum over all Nash Equilibria the ratio of the optimal social welfare to Nash Equilibrium social welfare, i.e.*

$$PoA = \max_{NE} \frac{SW(OPT)}{SW(NE)}$$

Furthermore, the Price of Anarchy can also be defined for a class of games as follows:

**Definition 3.** *The Price of Anarchy (PoA) of a class of Games ($\mathcal{G}$) the maximum over all games in $\mathcal{G}$ worst ratio of the optimal social welfare to Nash Equilibrium social welfare, i.e.*

$$PoA = \max_{G \in \mathcal{G}} \max_{NE(G)} \frac{SW(OPT_G)}{SW(NE_G)}$$

Let's go back to the market entry games. We show an example with $\frac{\text{SW(OPT)}}{\text{SW(NE)}} = \frac{5}{3}$. Can PoA be worse? Consider the $n$ player market entry game. In this case, SW(NE) $= n$, whereas SW(OPT) $\approx n + n - 1 = 2n - 1$. If we take the limit of $n \to \infty$, then

$$\frac{\text{SW(OPT)}}{\text{SW(NE)}} = 2 - \frac{1}{n} \xrightarrow{n \to \infty} 2.$$

So if $\mathcal{G}$ is the set of all market entry games, then PoA($\mathcal{G}$) $\geq 2$. However, is PoA($\mathcal{G}$) $> 2$, i.e does there exists a market entry game with PoA of greater than 2, or can we show that PoA of class of market entry games is at most 2?

**Question:** How do we show that PoA($\mathcal{G}$) $\leq 2$?
Consider an arbitrary entry game. Let $s^\star$ be optimal strategy profile. Let $s$ be a Nash Equilibrium profile (pure for now). We will show that SW($s$) $\geq \frac{1}{2}$SW($s^\star$).

Since $s$ is a NE

$$\forall\ s'_i\ :\ u_i(s) \geq u_i(s'_i, s_{-i})$$

**Question:** How can we use this property to achieve our goal?
Given that $s$ is a NE, no player $i$ wants to deviate from $s_i$ to the action $s^\star_i$ in the strategy profile $s^\star$ which gives optimal social welfare, i.e.

$$u_i(s) \geq u_i(s^\star_i, s_{-i}) = \frac{v_{s^\star_i}}{n_{s^\star_i}(s^\star_i, s_{-i})} \geq v_{s^\star_i} \mathbb{1} \left\{ s^\star_i \notin \bigcup_{k \neq i} \{s_k\} \right\}. \tag{1}$$

For any set $S$ of markets $S$, let $V(S)$ denote the total market value of all the markets in set $S$, i.e $V(S) = \sum_{j \in S} v_j$.

Then, the Nash Equilibrium social welfare is given by SW($s$) $= V\left(\bigcup_i \{s_i\}\right)$.

$$\begin{aligned}
\therefore \sum_i u_i(s) &\geq \sum_i u_i(s^\star_i, s_{-i}) && \text{by Nash Eq. condition} \\
&\geq \sum_i v_{s^\star_i} \mathbb{1} \left\{ s^\star_i \notin \bigcup_{k \neq i} \{s_k\} \right\} && \text{by (1)} \\
&\geq \sum_i v_{s^\star_i} \mathbb{1} \left\{ s^\star_i \notin \bigcup_{k} \{s_k\} \right\} && \text{if } A \subseteq B \text{ then } \mathbb{1}\{j \notin A\} \geq \mathbb{1}\{j \notin B\} \\
&\geq \underbrace{\sum_i v_{s^\star_i} \mathbb{1} \left\{ s^\star_i \notin \left(\bigcup_{k} \{s_k\}\right) \bigcup \left(\bigcup_{k<i} \{s^\star_k\}\right) \right\}}_{\text{The extra value of } s^\star \text{ on top of } s} && \text{same logic as the line above} \\
&= V\left(\left(\bigcup_k \{s_k\}\right) \bigcup \left(\bigcup_k \{s^\star_k\}\right)\right) - V\left(\bigcup_k \{s_k\}\right) && \text{(Valuation of all markets in } s^\star \text{ but not in } s) \\
&\geq V\left(\bigcup_k \{s^\star_k\}\right) - V\left(\bigcup_k \{s_k\}\right) && (V(\text{markets unique to } s^\star) - V(\text{markets unique to } s)) \\
&= \sum_i u_i(s^\star) - \sum_i u_i(s) \\
&= \text{SW}(s^\star) - \text{SW}(s) \\
\implies \text{SW}(s) &\geq \text{SW}(s^\star) - \text{SW}(s)
\end{aligned}$$

Hence, $\frac{\text{SW}(s^\star)}{\text{SW}(s)} \leq 2$.

## 1.2 Smoothness and Price of Anarchy of No-Regret Outcomes

We saw that we can upper bound the social welfare of NE to be half that of optimal social welfare. Can we say something similar for social welfare of some other strategy profiles? This motivates our next question.
**Question:** What can we say about social welfare when players play strategies learned by no-regret learning algorithms?
If such no-regret learning algorithm exists for each player $i$, then we will need to show that

$$\frac{1}{T}\sum_t u_i(s^t) \geq \frac{1}{T}\sum_t u_i(s_i', s_{-i}^t) - \epsilon(T).$$

**Question:** Furthermore, can we show that the social welfare of no-regret strategies converge to optimal social welfare?
That is, can we show

$$\frac{1}{T}\mathrm{SW}(s^t) \geq \frac{1}{T}\sum_t \mathrm{SW}(\mathrm{OPT}) - \epsilon(T) \quad ?$$

Observe that we showed, if $s$ is a NE, then

$$\sum_i u_i(s_i^\star, s_{-i}) \geq \sum_i u_i(s^\star) - \sum_i u_i(s).$$

(Note that this also holds true even if $s$ is not a NE.) Then, we invoked the definition of Nash Equilibrium and said

$$\sum_i u_i(s) \geq \sum_i u_i(s_i^\star, s_{-i}) \geq \sum_i u_i(s^\star) - \sum_i u_i(s) \qquad \square$$

**Definition 4.** *[$(\lambda,\mu)$ - smooth game] A game is $(\lambda,\mu)$ - smooth if $\forall\ s \in \Sigma_1 \times \cdots \times \Sigma_n$ and for $s^\star$ being optimal:*

$$\sum_i u_i(s_i^\star, s_{-i}) \geq \lambda \sum_i u_i(s^\star) - \mu \sum_i (s).$$

An interesting result about $(\lambda,\mu)$ - smooth games is as follows.

**Theorem 1.** *If game is $(\lambda,\mu)$ - smooth then*

$$SW(NE) \geq \frac{\lambda}{1+\mu} SW(OPT)$$

$$\implies PoA \leq \frac{1+\mu}{\lambda}$$

But, now that we decoupled the proof we can actually show a stronger statement.

**Theorem 2.** *If game is $(\lambda,\mu)$ - smooth and players use no-regret:*

$$\frac{1}{T} SW(s^t) \geq \frac{\lambda}{1+\mu} SW(OPT) - \epsilon'(T)$$

*where*

$$\lim_{T\to\infty} \epsilon'(T) \to 0$$

**Proof.** By no-regret w.r.t. $s_i^\star$

$$\begin{aligned}
\frac{1}{T}\sum_t \mathrm{SW}(s^t) &= \sum_i \frac{1}{T}\sum_t u_i(s^t) \\
&\geq \sum_i \left[\left(\frac{1}{T}\sum_t u_i(s_i^\star, s_{-i}^t)\right) - \epsilon(T)\right] \\
&= \left(\frac{1}{T}\sum_t\sum_i u_i(s_i^\star, s_{-i}^t)\right) - \sum_i \epsilon(T) \\
&\geq \frac{1}{T}\sum_t \left(\lambda \sum_i u_i(s^\star) - \mu \sum_i u_i(s^t)\right) - \sum_i \epsilon(T) \\
&= \lambda \mathrm{SW}(\mathrm{OPT}) - \mu \frac{1}{T}\sum_t \mathrm{SW}(s^t) - \sum_i \epsilon(T)
\end{aligned}$$

Hence,

$$\frac{1}{T}\text{SW}(s^t) \geq \frac{\lambda}{1+\mu}\text{SW}(\text{OPT}) - \epsilon'(T)$$

□

Many classes of games have been shown to be $(\lambda, \mu)-$ smooth ranging

- Routing games - Self-interested players route traffic through a congested network [1].
- Facility location games - Multiple utilities try to place facilities such that it reduces distance to clients and the number of facilities placed [2].
- Auction games: Generalized Sponsored Search auction, used for selling billions of dollars worth of ad slots in Google ads.

For instance, market entry games are a special case of the following Valid Utility Game.

Consider a (finite) game with $n$ players, where player $i$'s set of possible strategies is given by $\Sigma_i \subseteq 2^m$, where $m$ is the number of markets. The social welfare of a strategy profile $s$ is given by $SW(s) = V(s_1 \bigcup \cdots \bigcup s_n)$, where $V$ is a monotone submodular function.

**Definition 5.** *If $\Omega$ is a finite set, a submodular function $V : 2^\Omega \to \mathbb{R}$ satisfies*

- *For every $x, S, T$ such that $T \subseteq S \subseteq \Omega$ and $x \notin T$,*

$$V(S \bigcup \{x\}) - V(S) \geq V(T \bigcup \{x\}) - V(T)$$

A submodular function $V$ is monotone if for every $T \subseteq S$, $V(T) \leq V(S)$. Another way to interpret this is that $V(\cdot)$ is monotone submodular function if $V(S \bigcup T) - V(T)$ is decreasing in T.

$$u_i(s) \geq \underbrace{V\left(\bigcup_j \{s_j\}\right) - V\left(\bigcup_{j \neq i} \{s_j\}\right)}_{\text{marginal contribution of } s_i \text{ to value function}}$$

**Theorem 3.** *Valid utility games are (1,1) - smooth.*

Proof is identical to market entry games. Just do pattern matching to generalize.

# References


[1] Tim Roughgarden. Chapter 18 - routing games.

[2] Eva Tardos. Facility location game.