Information Asymmetries in Common-Value Auctions with Discrete Signals

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Abstract

We study the role of information asymmetries in common-value auctions. We consider a common-value hybrid auction (containing first-price auctions as a special case and second-price auctions in the limit) among two bidders, in which the highest bidder wins and pays his bid with probability $\kappa > 0$, and the second-highest bid with the remaining probability. Prior to the auction, each bidder receives a signal from some discrete and bidder-specific signal space. We assume that the signals of the two bidders are *informative* and *affiliated*, but otherwise are drawn from some arbitrary and asymmetric joint distribution.

Our main result is an explicit characterization of the (unique) equilibrium of this common-value hybrid auction, based on a simple recurrence relation. This explicit characterization allows us to derive several qualitative insights: (1) The interim utilities of the bidders are non-decreasing as a function of κ; hence the revenue is non-increasing. (2) We characterize the limit equilibrium of the second-price auction selected by the process of taking $\kappa \to 0$; we show that it entails extensive free-riding, but avoids the equilibrium collapse predicted by alternative equilibrium selection of Abraham et al. and of Einy et al. (3) In a first-price auction with asymmetrically informed bidders, the Linkage Principle can fail to hold even when signals are binary: Public revelation of a signal to both bidders may decrease the auctioneer's revenue. (4) Publicly acquired additional information has surprising effects; for example, it can result in strictly lower utility for a bidder, or a bidder may prefer his competitor to receive extra information.

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1 Introduction

Information is at the heart of much of the modern economy. Entire industries are concerned with the collection, processing, and evaluation of information in a wide variety of settings. From a theoretical viewpoint, the acquisition of information is frequently interpreted as obtaining a signal about the state of the world; the state of the world determines the payoff of available actions, and as a result, more information allows an agent to choose actions with higher payoff (see, e.g., [23, 33, 2]). This framework is very general; as a result, it is often hard to draw clear quantitative conclusions about the value of information.

Common-value auctions with asymmetrically informed bidders provide a natural domain in which to study the value of information. Indeed, obtaining a complete understanding of asymmetric common-value auctions has long been a research goal, towards which only partial progress had been made. In a common-value auction, the state of the world is simply the object's value, and the bidders' signals reveal partial information about this value. Much of the early foundational literature has focused on two very special cases: (1) bidders are symmetric in the sense that their signals are drawn from the same distribution (e.g., [28]), or (2) one bidder has access to all the information another bidder has, plus additional information [40, 29, 12]. However, in order to fully understand the value of information in this context, it is crucial to consider fully general asymmetric information structures between the bidders.

In the present article, we focus on common-value auctions in which bidders get *discrete* signals (such as binary signals, or signals with a few discrete values). Continuous signal spaces naturally model domains in which the bidders receive fine-grained estimates of the item's value. On the other hand, discrete signal spaces are more suitable when the signals are very coarse-grained. Natural applications are the outcomes of a small number of tests that could yield positive or negative results (e.g., in evaluating a potentially oil-rich tract of land), or the use of cookies in Internet advertising. In the context of Internet advertising, the value of an advertisement opportunity has a strong common-value component (e.g., whether a visitor to a web site is a human or a bot contributes strongly to the value of all advertisers), and the signals may naturally correspond to binary information characterizing whether the visitor recently purchased from or visited certain web sites. Many companies have access to cookies that store such information. Estimating the value of such cookies is rapidly becoming more important due to the emergence of Data Management Platforms [37, 7, 30] such as BlueKai or Brighttag, where third-party information brokers sell information to help advertisers determine whether or how to bid for a particular site visit.

Prior work on asymmetric common-value auctions has only been able to characterize the equilibria for very specific information structures [4, 16, 25, 12]. Parreiras [32] characterized the (unique) equilibria in two-bidder common-value auctions with continuous signal spaces (with atomless distributions) in terms of an implicit function and differential equation. Such continuous signal spaces give rise to deterministic equilibria. For discrete signal spaces, equilibria will typically be mixed. Even the equilibria of general asymmetric *binary* signals had only been characterized in very special cases (such as when the low signal implies a negative expected value for the object [4]).

The main contribution of the present work is a complete characterization of the (unique) mixed Nash equilibrium of common-value auctions with two bidders and essentially arbitrary discrete signal structures. The only assumptions we make on the signals are that they are *informative* (a higher signal implies a higher expected value of the item) and *affiliated* (a higher signal implies a stochastically higher signal distribution for the other bidder). We characterize the equilibria of all hybrid auctions in which the winning bidder pays his bid with some fixed positive probability κ and the second-highest bid with the remaining probability. Such auctions include the first-price auction and, in the limit, the second-price auction. Our work thus simultaneously generalizes essentially all prior work on common-value auctions with asymmetric bidders and discrete signals [4, 16, 25, 12].

The equilibrium characterization is patently constructive. Its main component is a recurrence relation describing the supports of the equilibrium distributions for pairs of signals of the bidders. The constructive nature immediately implies an algorithm for explicitly computing the bidders' equilibrium strategies, with running time linear in the size of the signal spaces. More importantly, the clean characterization allows us to perform several comparative statics analyses and derive insights that have eluded prior work.

First, it makes it easy to compute the limit distribution as the auction converges to a second-price auction. This limit process provides a natural candidate for equilibrium selection among the well-known multiplicity of equilibria for second-price auctions. The explicit characterization of this limit equilibrium allows us to observe that contrary to other proposed equilibrium selection mechanisms, free-riding behavior persists in the limit. The selected equilibrium has high revenue, in stark contrast to alternative equilibrium selection proposals of Abraham et al. [1] and Einy et al. [10] which may lead to revenue collapse.

Second, it allows us to easily perform a complete revenue ranking of all hybrid auctions. We show that interim bidder utility conditional on any signal instantiation is a non-decreasing function of κ, the probability that the winner pays his bid. Hence, the first-price auction is preferred by bidders, and the limit equilibrium of the second-price auction is preferred by the auctioneer. Our result is in line with the result of Milgrom and Weber [28] for the symmetric and continuous signal setting. Parreiras [32], for continuous signal spaces, showed the weaker result that the revenue of any hybrid auction is no less than the revenue of the first-price auction. A complete revenue ranking was not obtained and left as an open question; the clean nature of the equilibrium characterization allows us to resolve the open question for the case of discrete signals.

Third, the equilibrium characterization allows us to give a precise condition when asymmetric signal structures lead to symmetric equilibria. A sufficient condition had been exhibited by Hausch [16]. We give a condition that is both necessary and sufficient; furthermore, it has a natural interpretation in terms of reverse Hazard rates.

The ease of explicitly computing equilibrium strategies and associated bidder utilities also allows us to explore the space of auctions and their solutions more systematically, and to discover several surprising phenomena. Perhaps most surprisingly, we show that the linkage principle fails to hold in the presence of asymmetric signals. In fact, this occurs even in the very simple scenario of a first-price auction where bidders receive asymmetric binary signals that are independent conditional on the true value. We explicitly provide an instance in which the auctioneer's expected revenue *decreases* if an extra binary signal is commonly revealed to both bidders. This answers an open question explicitly asked by Parreiras [32].

The failure of the linkage principle implies that information can have positive value to both bidders *even when it is revealed to both.* Another surprising phenomenon is that in some cases, information is valuable to both bidders *only when revealed to bidder 1.* In other words, both bidders strictly increase their ex ante expected utility when a signal is revealed to bidder 1, but neither bidder increases his utility when the signal is revealed to bidder 2 (or to both). Such an alignment of incentives between both bidders is surprising in that one would expect information in auctions to carry negative externalities. In fact, positive externalities can already occur in the same very restrictive setting of binary values and signals described above. There are instances in which both bidders prefer the previously more informed bidder to become even more informed, and other instances in which both bidders prefer the previously less informed bidder to become equally informed to the other one.

Model and Results

We consider a common-value hybrid auction in which the item being auctioned has the same value $v \in \mathbb{R}_+$ to all bidders; v is drawn from a known distribution. The winner of the auction pays his bid with probability $\kappa > 0$, and the second highest bid with the remaining probability. When $\kappa = 1$, we obtain the standard first-price auction, while the limit $\kappa \to 0$ provides a unique selection process among the multiple equilibria of the second-price auction. Each bidder receives a (usually imperfect) signal about the value from a finite set of possible signals; the signal spaces or distributions of different bidders need not be the same.

We assume that the signals are informative of the actual value (a higher signal implies a higher expected value) and that the signals of the bidders are affiliated (a higher signal for one bidder implies a stochastically higher signal for the opponent). In addition, in order to establish uniqueness of the equilibrium, we assume that the joint signal distribution has full support (i.e., every signal of the opponent has non-zero probability conditional on a signal); the assumption is not needed to show that our construction yields a valid equilibrium.

Uniqueness and Characterization of Equilibrium. Our main result (in Section 4) is an explicit characterization of the (unique) mixed Nash equilibrium of the hybrid common-value auction with two bidders. The key part of the characterization is a clean recurrence relation among the supports of the bidders' distributions under different signals. The recurrence relation gives rise to a simple iterative algorithm to compute this equilibrium as well as the resulting value of the auction for each player. The distributions characterized by the recurrence give rise to an equilibrium whenever the signals are informative and satisfy a reverse hazard rate dominance monotonicity condition; this is a weaker requirement than affiliation or likelihood ratio dominance. Even under these weak conditions, the equilibrium thus characterized is unique among monotone equilibria; it is the overall unique equilibrium when the signals are affiliated and have full support.

Our equilibrium analysis extends the monotonicity characterization of Rodriguez [34] to show that the equilibrium supports of each bidder must be consecutive intervals falling into a common region $[\underline{b}, \overline{b}]$, where $\underline{b}$ must equal the expected value conditional on both bidders receiving their lowest possible signal. Therefore, this region can be divided into consecutive intervals $I_{y,z}$, corresponding to the bidders' signals y, z. Our construction can be divided into two phases. In the first phase, we start from the rightmost interval, which must be associated with the highest signals of the two bidders. Using the indifference of each bidder among bids in the support of his equilibrium, we establish the form of the cumulative density functions in that interval. Then, we recursively establish which signals must be associated with the left neighboring interval. The surprising and remarkably useful insight here is that although the exact form of the equilibrium CDF depends on the actual value of the upper bound of the interval, deciding which of the two CDFs will become zero first is *independent* of it. The latter comparison depends on the joint signal distribution only through the conditional reverse hazard rates (i.e., the reverse hazard rate of the other bidder's signal distribution conditional on a signal). Hence, we can recursively characterize which signals correspond to the neighboring interval. Once the entire interval structure has been established, the condition on the lowest interval determines the exact upper and lower bounds of these intervals.

The independence of the interval structure on any parameters of the distribution except the conditional reverse hazard rates is complemented by an even more striking independence: the interval structure, as well as the probability mass that each bidder allocates within each interval, is independent of κ, the probability that a winning bidder pays his bid. Hence, it is the same for any hybrid auction, which we prove by introducing a tying function for the discrete setting and establishing its invariance under κ. An analogous result was shown by Parreiras [32] in the continuous version.

A very restricted special case of our recurrence construction was given by Malueg and Orzach [25] only for the case of a first-price auction under the following restricted discrete information structure: conditioned on a true value v of the item, the signals y_v and z_v are completely determined. Furthermore, if $y_v = y_{v'}$ for $v < v'$, then $y_v = y_{v''}$ for all $v \leq v'' \leq v'$; in other words, signals partition the space of values into contiguous intervals. This information structure can be easily seen to satisfy the informativeness and affiliation property. Moreover, in this special case, the interval structure of the equilibrium can is known explicitly; this dramatically simplifies the equilibrium construction. Unfortunately, such an information structure is too limited to be able to capture even the general affiliated binary signal setting, which we believe to be of great importance.

Second-Price Equilibrium Selection. The uniqueness of the equilibrium for all hybrid auctions implies that the limit of the equilibrium strategies as $\kappa \to 0$ selects a unique equilibrium for the second-price auction (which is well-known to have a multiplicity of equilibria). This equilibrium will still be mixed, but will have finite support. Specifically, as κ decreases, we show that the CDF within each interval of the equilibrium characterization become more and more convex, converging to a point mass at the upper bound of the interval as $\kappa \to 0$. In addition, the upper bound of each interval $I_{y,z}$ converges to the expected value conditional on the two signals y, z. Thus, the limit equilibrium of the second-price auction involves randomization over expected common values conditional on pairs of signals that are likely to occur together.

An interesting instantiation of this equilibrium selection is the case when only one bidder is informed. We show (in Section 6) that the limit equilibrium of this setting involves extensive free-riding. More specifically, the uninformed bidder will bid so as to simulate the informed bidder's distribution in the sense that the ex-ante bid distributions are symmetric. For instance, when the informed bidder receives a binary signal, the uninformed bidder will bid the expected common value conditional on the high signal with the probability that the informed bidder receives the high signal, and the common value conditional on the low signal with the remaining probability.

More interestingly, in this setting we show that any hybrid auction is revenue and utility equivalent, which is a corner case of our complete revenue ranking theorem. This equivalence stands in stark contrast to alternative equilibrium selection processes: both the tremble-robust equilibrium of Abraham et al. [1] and the iterative elimination of undominated strategies with a Pareto-dominant refinement of Einy et al. [10] would select an equilibrium without free-riding, in which the uninformed bidder always bids the lowest expected value.

The main conceptual difference between our equilibrium selection process and those put forth by Abraham et al. and Einy et al. is that when bidders believe that the first-price auction might be run with positive probability, then even the informed bidder fears overbidding; hence, it is natural that he will slightly shade his bid. Given this shading, the uninformed bidder can potentially gain utility by free-riding (though at equilibrium he will still get zero utility). In contrast, the random reserve aspect of the tremble-robust equilibrium does not entail such a fear of overbidding from the informed bidder.

Complete Revenue Ranking. In our general setting, we show (in Section 5) that the auctioneer's expected revenue at equilibrium is a non-increasing function of κ, the probability that the winner pays his bid. Therefore, every κ-hybrid auction revenue-dominates a κ'-hybrid auction for any $\kappa < \kappa'$, and the limit equilibrium of the second-price auction yields revenue no lower than any hybrid auction, including a first-price auction. The main technical challenge in proving the revenue ranking result is the fact that there are two opposing forces at play when decreasing κ. The equilibrium bid distributions stochastically increase, while the expected price gives more weight to the expected second highest bid rather than the expected highest bid. We show that the net effect of these two opposing forces on the expected price is always positive, and thus the bidders' interim utilities decrease as κ decreases.

Reverse Hazard Rates and Symmetric Equilibrium Condition. As we pointed out earlier, the support structure of the equilibrium is uniquely defined by the reverse hazard rates of the conditional signal distributions. The corresponding characterization naturally yields a necessary and sufficient condition for the equilibrium to be symmetric. We show that the following condition is necessary and sufficient for obtaining a symmetric equilibrium: (1) the signal sets of the two bidders have the same size, and (2) for each signal σ, the conditional reverse hazard rate of Z's signal σ given that Y has signal σ is the same as the reverse hazard rate of Y's signal σ given that Z has signal σ.

For binary signals, this is equivalent to stating that both bidders receive the high signal with equal probability. The latter condition for binary signal settings was also observed by Banerjee [4], but under the restriction that bidders do not bid when receiving their low signal. For general discrete signal settings, Hausch [16] proposed a much more restrictive sufficient condition for the equilibrium to be symmetric; the condition yields almost symmetric distributions. Our necessary and sufficient condition significantly relaxes the condition of Hausch [16], and completely settles the characterization of symmetric equilibria.

Failure of the Linkage Principle in First-Price Auctions. Our constructive equilibrium characterization allows us to explicitly compute the effects of revealing additional information, with respect to the expected revenue and bidder utility. Several unexpected phenomena arise.

In the symmetric information setting, the well-known Linkage Principle result of Milgrom and Weber [28] implies that in a common-value auction, it is always profitable for the seller to follow an information-revealing policy. We show (in Section 8) that in a common-value setting with *asymmetric* discrete private information, the Linkage Principle may fail to hold, even for first-price auctions. Specifically, we exhibit a simple example in which a policy of revealing an independent signal to two asymmetrically informed bidders lowers the auctioneer's revenue.

The intuition is as follows: if the bidders' signals are asymmetric, then the revelation of an additional, independent, signal reveals different amount of information to both bidders; the conditional random signal distributions are affected very asymmetrically. For example, an additional highly informative signal about the common value is more correlated with an accurate previous signal than an inaccurate one. Therefore, the relative quality of the two signals after the revelation can potentially change, leading to a different equilibrium structure.

The key technical step in unveiling the failure of the Linkage Principle is that with ex-ante asymmetrically informed bidders, the expected total utility of the two bidders is not a concave function of the prior on the common value. The public revelation of an additional signal can be cast as a randomization between two prior distributions, such that the initial prior can be expressed as a convex combination of the two. The non-concavity (and hence non-convexity of total revenue) implies that the expected utility under this randomization is not necessarily lower than the expected utility under the initial prior. We provide a concrete example with these properties.

Value of Additional Information. The failure of the Linkage Principle constitutes a first surprising example about the value of information in common-value auctions: it shows that sometimes, a piece of information increases the bidders' total utility even when released to both of them. In Section 9, we undertake a more comprehensive exploration of unexpected properties of the value of information. We observe surprising non-monotonicity properties and positive externalities, even in the following drastically simplified special case of the auction model: the value v of the item and the signals of the bidders are all binary, and all signals are independent conditional on v.

We analyze the effect on bidder utility from revealing another binary signal, which is also independent conditional on v. Our motivation partly derives from the emerging data market in online advertising, where the additional binary signal can be thought of as an extra cookie that is owned by a third-party information seller. Hence, our results can have managerial implications on monetizing such extra information.

Using our general characterization, and its specialization to the binary case, we observe a number of surprising properties of the value of information. First, analogous to observations under slightly different models made by Engelbrecht-Wiggans and by Kessler [11, 19], we show that additional information, publicly obtained, may *decrease* a bidder's utility. Thus, a bidder may want to remain *strategically ignorant* about the value of the item.

More surprisingly, we exhibit settings in which both bidders *agree* on which bidder should receive the signal; in other words, both bidders' utilities are individually maximized if and only if one particular bidder receives the signal. There are two varieties of this phenomenon: A poorly informed bidder may prefer an extremely well-informed opponent when the item is likely to be of high value. On the other hand, when the item is extremely likely to be of high value, a very well-informed bidder may prefer an equally informed opponent over a poorly informed one.

2 Related Work

Our work naturally touches on several separate areas: common-value auctions, the linkage principle, the comparison of auction formats, and the value of information in auctions and games more generally. We discuss these three related areas separately.

Common-Value Auctions. Common-value auctions are a classic field of study within economics. Rigorous mathematical analysis appears to have been initiated by Wilson [40, 41, 42] and the Ph.D. thesis of Ortega-Reichert [31]. Subsequently, the breadth and depth of analysis were significantly expanded in Milgrom's Ph.D. thesis [27], and the influential work by Milgrom and Weber [29, 28], and Engelbrecht-Wiggans et al. [12]. (A comprehensive overview is given by Kagel and Levin [17], albeit with a focus on experimental work.) Unfortunately, most common-value auction models are notoriously difficult to study, in particular in asymmetric settings. Therefore, much of the literature assumes that agents are symmetric with respect to the information they receive about the value of the item (see, e.g., [42, 28, 12, 20]).

Asymmetric Common-Value Auctions. Much less is known for common-value auctions with asymmetrically informed bidders. Here, what little is known tends to be restricted to the case of two bidders. Early classic articles by Wilson, Milgrom, Weber and Engelbrecht-Wiggans [40, 29, 12] (see also [20]) consider only the case in which one bidder is perfectly informed about the value of the item, while the other bidders are entirely uninformed. Engelbrecht-Wiggans [11] exhibits a scenario in which the uninformed bidder actually prefers remaining uninformed over becoming perfectly informed. In other words, a better signal may result in less value for a bidder.

Several other pieces of work [27, 35, 36, 22, 16] make partial progress in characterizing equilibria for very specific classes of asymmetric common-value auctions with two bidders. Rothkopf [35] (Erratum [36]) analyzes very specific distributions in two-bidder auctions, and also shows that under restrictive assumptions, there exist equilibria in which bidders simply scale their signal by a constant. Laskowski and Slonim [22] explicitly calculate equilibria under a severely reduced strategy space, namely, when bidders are restricted to adding (or subtracting) a constant value to the signal they receive. Banerjee [4] characterizes the (unique) equilibrium of the first-price auction with conditionally independent binary signals, under the strong assumption that the lower possible value of the item is so negative that when receiving a low signal, a bidder will never submit a bid for the item.[1]

Hausch [16] considers a setting in which two bidders receive signals from a discrete finite set. He requires that the conditional distribution of bidder 1's signal, given any signal of σ for bidder 2, be the same as the conditional distribution of bidder 2's signal, given a signal of σ for bidder 1. While this assumption is more general than symmetry among the bidders' signals, it is only slightly more so, and indeed, Hausch shows the existence of a symmetric equilibrium in this case. Our necessary and sufficient condition for a symmetric equilibrium heavily generalizes the condition of Hausch. However, generally, the models in the above works are restrictive (making bidders almost symmetric), or the bidders' behaviors are severely restricted.

Malueg and Orzach [25] consider a two-bidder first-price auction with a special discrete and asymmetric information structure: the common value is drawn from a discrete set of values, and each bidder's signals partition of the set of possible values. The partition corresponds to consecutive intervals of the set of possible values, and the partitions of the two different bidders must be overlapping, but not necessarily the same. This restrictive class is not general enough to model arbitrary conditionally independent binary signals; for binary signals, the condition implies that one bidder's low signal deterministically reveals the lowest possible item valuation, while the other bidder's high signal deterministically reveals the highest possible valuation. Malueg and Orzach give a recursive process for computing an equilibrium in this setting, similar to the one proposed here. In fact, it is easy to see that their signal structure satisfies our informativeness and affiliation

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[1] A possible motivation of this assumption is costly exploration of an oil tract which would be required prior to discovering that it yields no oil.

properties (the right-hand side of the affiliation inequality (1) is always 0); hence, their recursive process is a special case of our general equilibrium characterization.

Siegel [38] studies all-pay auctions with discrete signals and also gives a recursive construction for the unique equilibrium, which is also monotone. The assumptions in [38] are different and orthogonal to the affiliation and informativeness assumptions that we make. In addition, the recurrence in [38] is qualitatively different from ours: in all-pay auctions, the cumulative density functions of the mixed strategy conditional on any signal are piecewise linear (i.e., the density is piecewise constant), while for hybrid auctions, we show that the density functions are piecewise convex.

Several articles study more general affiliated signal models. Wilson [41] studies a common-value first-price auction under a general model of asymmetric continuous signals independent conditional on the item's value. Wilson claims (without proof) that equilibria will be monotone, and gives a set of differential equations characterizing an equilibrium. For a more general asymmetric setting in which the bidders' signals can be affiliated, Rodriguez [34] proves monotonicity of bidding strategies (i.e., higher signals imply higher bids — more formally captured in Lemma 2). Rodriguez uses this monotonicity result to show uniqueness of the equilibrium with symmetric bidders. Our work uses this characterization as a starting point and proves uniqueness and existence of an equilibrium in an arbitrary asymmetric setting with affiliated signals. Parreiras [32] considers a two-bidder common-value hybrid auction with continuous signal distributions. He shows that there exists a unique equilibrium, which is pure. The continuity of the signal distributions (and with it the continuity of the expected value as a function of the signal) makes the analysis in [32] qualitatively very different from ours, allowing the use of tools from real analysis. For example, the equilibrium in our discrete setting is necessarily mixed. More importantly, the structural results of our analysis can be leveraged for a deeper understanding of equilibrium properties: for example, they allow us to completely rank revenue of hybrid auctions, observe the failure of the linkage principle, and derive necessary and sufficient condition for the existence of a symmetric equilibrium.

Very little is known for asymmetric common-value auctions with more than two bidders. A natural generalization of our model (in which each agent receives a signal from a discrete ordered set of signals) to multiple bidders is studied by Wang [39], who analyzes equilibria of first-price auctions. However, Wang's analysis is restricted to symmetric bidders; furthermore, he only studies signals that are independent conditional on the true value of the item. McAdams [26] extends the monotonicity characterization of Rodriguez [34] to n bidders with continuous and affiliated signal distributions; among others, he shows that the bidding strategies under any equilibrium of a common-value first-price auction are monotone in the signal. The analysis in [26] also easily extends to hybrid auctions with $\kappa > 0$.

Equilibrium Selection in Second-Price Auctions and Revenue Comparisons. It is well known that the second-price auction has a multiplicity of equilibria in common-value settings, and several articles have proposed equilibrium refinements. Most recently, Abraham et al. [1] propose the notion of the "tremble-robust equilibrium;" they perturb the auction by having an external bidder impose a random reserve on the initial set of bidders, to help select among the multiple equilibria of the second-price auction. They mainly analyze two cases: (1) one informed bidder, and (2) two bidders with asymmetric binary signals. They show uniqueness of the equilibrium under their refinement in these two settings. One of their main findings is that under this refinement (unlike the revenue ranking result of Milgrom and Weber [28] for the symmetric case), the revenue of a second-price auction can collapse when one bidder is completely uninformed.

A similar reversal of the revenue ranking result of Milgrom and Weber [28] was observed by Malueg and Orzach [24], using an equilibrium selection proposed by Einy et al. [10], and by Cheng and Tan [6] who use a private value perturbation to select a unique equilibrium. Specifically, Cheng and Tan [6] consider private value perturbations in which the private value is perfectly correlated with the common value, and the perturbation is symmetric among bidders. They show uniqueness of the selected equilibrium; under the selected equilibrium, the revenue of the second-price auction is at least as much as the revenue of the first-price auction. Larson [21] shows that this result is heavily dependent on the perfect correlation and the symmetry of the perturbation; it fails to hold under more general perturbations.

In our work, similar to Parreiras [32], we use limits of hybrid auctions as a way of selecting an equilibrium of the second-price auction. Whenever the winning bidder pays his bid with positive probability κ, the equilibrium of the hybrid auction is unique. Surprisingly, the equilibrium selected as $\kappa \to 0$ is qualitatively very different from the one selected by the tremble-robust equilibrium refinement. For instance, our revenue ranking result shows that this equilibrium achieves at least as much revenue as any hybrid auction and therefore at least that of the first-price auction. This also implies that the revenue of the second-price auction cannot collapse under this equilibrium selection.

Limits of the Linkage Principle. The classic work of Milgrom and Weber [28] introduced the now well-known linkage principle in auctions, stating that in a symmetric common-value setting, the more information is linked to the price of the winner, the higher the revenue of that auction format. This principle implies that in a symmetric setting, the second-price auction yields higher revenue than the first-price auction, and that the auctioneer always increases his revenue by following the policy of revealing all his information.

We show here that the linkage principle fails when bidders are ex ante asymmetrically informed, even in the very simple setting of a first price auction in which bidders receive binary signals. The failure of the linkage principle is well known for private-value settings [20] and has also been shown under some other extensions of the symmetric single-item common value setting. For instance, Fang and Parreiras [14] show that it fails when bidders have budget constraints, and Foucault and Lovo [15] show that it fails when the signals are multi-dimensional. Our results show that simply breaking the symmetry in the ex ante information causes the failure of the linkage principle even in very simple binary signal common-value settings.

Campell and Levin [5] analyze a special case of a two-bidder asymmetric common-value auction: both the signals and the value of the object are binary, and the prior on the value is uniform over $\{0, 1\}$, while the signals have symmetric error. This is a special case of the binary signal setting that we analyze in Section 7. They show that under their restricted asymmetric setting, the public release of information always increases revenue and hence agrees with the linkage principle intuition. Our results show that this is a product of their special information structure; it does not even hold generally in a setting of binary signals and value. They also show an example in which a bidder's utility might decrease with extra information. In Section 9, we give a region of the binary signal setting where such an effect can be observed; we complement it with several other unintuitive outcomes that might result from the acquisition of extra information.

Hausch [16] analyzes an example with asymmetric binary signals in which, unlike the intuition implied by the linkage principle, making bidders more symmetric can actually decrease revenue. In his example, the common value is a priori either 0 or 1 with equal probability, and one bidder receives an informative binary signal with symmetric error (see Section 7), while the other bidder receives an uninformative one. Hausch shows that as the uninformed bidder's signal quality increases and approaches the informed one's, the revenue of the auction does not monotonically increase. However, the latter example does not show that the public revelation of a signal by the seller can decrease the revenue, which is what we show. His approach was limited by the fact that his analysis could only be applied in cases where the prior probability of the value being 0 or 1 is equal. In order to produce our example, where the public revelation of a signal can actually decrease the revenue, one has to characterize equilibria with non-uniform prior probabilities. This is enabled by our general equilibrium characterization.

The Value of Information. A more general investigation of the value of information in game-theoretic settings in general, and auctions in particular, has been undertaken in a sequence of articles [23, 33, 2]. Starting from the classic work of Karlin and Rubin [18] on statistical tests, this line of work treats signals in economic settings as statistical tests about a state of the world (e.g., the value of an item in an auction).

The model is typically that based on the signal, the owner of the signal wants to choose a suitable action. The value of a signal should be judged based on the utility that is achieved as a result of the action taken. Compared to the original work of Lehmann [23], the work of Persico [33] and Athey and Levin [2] broadens the classes of utility functions and information structures whose value can be characterized.

A slightly related question was investigated by Eső and Szentes [13]. They assume a private-value auction model in which the auctioneer of the item has control over the signals the bidders receive (which may help them determine their valuations), but does not know the signals' values. [13] focuses on revenue-maximizing ways of releasing these signals.

Compte and Jehiel [8] and Persico [33] specifically study the value of information in auction settings. Compte and Jehiel [8] focus on a private value setting, where they show that dynamic auctions give bidders more incentive for (gradual) costly information acquisition than sealed-bid auctions, which require a one-time investment in information. Thus, an auctioneer's revenue is higher for dynamic auctions. Persico's model [33] is very general in terms of its information structures; hence, it does not let us characterize explicitly the equilibrium strategies or value of information in our specific auction scenario.

Recent work of Babaioff, Kleinberg and Paes Leme [3] considers mechanism design for selling information to one bidder. They consider only the information selling stage and examine how to maximize revenue. Our work explicitly models the subsequent use of the information in the second stage and considers the value of information when sold to more than one participant, resulting in externalities. Their focus is on information as a much more flexible product than items traditionally sold in auctions; for example, they show that the revelation principle may fail to hold. In contrast, we consider selling a signal only as a whole.

3 Preliminaries

We consider a common-value single-item auction scenario with two bidders, Y and Z. The item has a *common* value of $v \in \mathbb{R}_+$ to the two bidders. The realization of v is not observed by the bidders, who only receive (more or less informative) signals about v, as described below. This model fits the classical motivation of auctions for possibly oil-rich tracts of public land: at the time of the auction, it is unobserved how much oil the land provides, but the value is the same to the two bidders. A more recent example domain is the auctioning of an advertising slot on a webpage impression; so long as the advertisers are sufficiently comparable (e.g., Domino's vs. Pizza Hut), the expected value of the visitor will be the same (or sufficiently similar) to them.

Each bidder Y, Z receives a signal y, z that takes values in some finite and ordered *signal space* $\Sigma_Y = \{1, \ldots, k_Y\}$, $\Sigma_Z = \{1, \ldots, k_Z\}$. We denote with $\pi(v, y, z)$ the probability density function of the joint distribution from which the signals of the two bidders and the value of the item are drawn. We will denote with $\pi(y, z)$ the marginal joint density function of the signals, with $\pi_P(\sigma)$ the marginal density function of the signal of bidder $P \in \{Y, Z\}$ and with $\pi(\cdot|\sigma)$ the density of a bidder's signal conditional on a signal σ of the other bidder. We will denote with $V(y, z)$ the expected value of the item conditioned on bidder Y receiving signal y and bidder Z receiving signal z. In addition, we denote with $V_Y(y)$, $V_Z(z)$ the expected value conditioned on the signal of only one bidder. We will make the following standard assumptions on the information structure of the agents:

Assumption 1 *The signals are* informative *of the true value: $V(y, z)$ is strictly increasing in both arguments.*

Assumption 2 *The signals of the two bidders are* affiliated*: if $y \leq y'$ and $z \leq z'$, then*

$$\pi(y, z)\pi(y', z') \geq \pi(y', z)\pi(y, z'). \tag{1}$$

Assumption 3 *The joint distribution of signals $\pi(y, z)$ has* full support*: $\pi(y, z) > 0$ for all $y \in \Sigma_Y, z \in \Sigma_Z$.*

The notion of affiliation was introduced in the early work of Karlin and Rubin [18], and several properties were established by Milgrom [27] in the context of auctions. Essentially, affiliation states that the conditional distributions of the opponent's signals are completely ordered with respect to the monotone likelihood ratio stochastic order, as a function of a bidder's signal. One interesting property is that in order to be affiliated with each other, it is sufficient (though not necessary) for the two bidders' signals to be affiliated with the true value v and independent conditioned on v. This fact is captured by Lemma 19 in Appendix A for the case of discrete distributions, and had been previously known for the case of continuous distributions (see [9, p. 14]). In addition, when the value v is binary, informativeness of the signals suffices for them to be affiliated with the value. This fact is captured by Lemma 20 in Appendix A.

The bidders submit bids based on the signals they receive. The bidder with the higher bid wins the item. We analyze a general class of payment functions: hybrid auctions between the first- and second-price auction, in which the winner pays a convex combination of his bid and the second highest bid. Formally, the payment of the winner is $K(a, b) = \kappa a + (1 - \kappa)b$, where a is the winning bid, b the second-highest (losing) bid, and $\kappa \in (0, 1]$ a fixed parameter. When $\kappa = 1$, we recover the first-price auction as a special case, and in the limit $\kappa \to 0$, we obtain the second-price auction.

The strategy of a bidder P is a mapping from a signal σ in his signal space to a distribution F_σ^P over possible bids.

In the last part of this article, we will be particularly interested in the value of acquiring information in the form of better (or additional) signals, and the types of outcomes to be expected if such better signals are sold or auctioned by a self-interested seller. The seller possesses information in the form of a signal that is independent of the ones that the bidders previously possess. The seller sells her signal prior to observing its actual value. The seller's signal is a digital good in that it can be allocated to more than one bidder; the seller chooses whether to sell it to both bidders, exclusively to one bidder, or whether not to reveal the signal to either of them.

4 Equilibrium of the Common-Value Hybrid Auction

In this section, we characterize the bidding behavior of the two bidders at equilibrium of the single-item hybrid auction. The behavior will be randomized, as generally, even first-price auctions have no pure Nash Equilibria. The behavior also obviously depends on the signals the two bidders receive. Any equilibrium is characterized by the bidders' cumulative distribution functions F_y^Y, F_z^Z for all possible signals $y \in \Sigma_Y, z \in \Sigma_Z$. Our main result is the following theorem:

Theorem 1 (Uniqueness and Existence) *The single-item common-value hybrid auction has a unique mixed Nash Equilibrium whenever bidders have affiliated and informative discrete-valued signals and the joint distribution of signals has full support. Moreover, there is an algorithm whose running time is linear in the size of the signal spaces* $O(|\Sigma_Y| + |\Sigma_Z|)$*, and which explicitly computes the cumulative distribution functions* F_y^Y*,* F_z^Z *and the expected utilities of the bidders from the given parameters of the auction.*

The proof of the theorem starts with a characterization of equilibrium bids, based on a monotonicity lemma of Rodriguez [34]. (Rodriguez describes his results in a first-price auction setting, but his formulation is general enough to accommodate hybrid auctions as well.) Subsequently, we show that the characterization implies the existence of a unique solution, and we give a recursive way of finding this solution.

To express the characterization of the equilibrium and its analysis, we use the following notation: $u_\sigma^P(b)$ denotes the expected utility of bidder P when he receives the signal σ and bids b. W_σ^P is the support of the distribution F_σ^P, and $\underline{b}^P = \min_\sigma \inf W_\sigma^P, \overline{b}^P = \max_\sigma \sup W_\sigma^P$ are the highest and lowest bids in the supports of the two bidders' distributions.

Lemma 2 (Rodriguez [34]) *Suppose that the distributions F_σ^P constitute a mixed Nash Equilibrium of the common-value auction. Let $\underline{b} = \max\{\underline{b}^Y, \underline{b}^Z\}$. If Assumptions 1 and 2 hold, then:*

1. $\overline{b}^Y = \overline{b}^Z = \overline{b}$.
2. *The supports of the distributions F_σ^P are monotone increasing in the region $(\underline{b}, \overline{b}]$. That is, if $x \in W_\sigma^P, x' \in W_{\sigma'}^P$, and $\sigma > \sigma'$, then $x \geq x'$.*
3. *$\sum_{\sigma \in \Sigma_P} F_\sigma^P(b)$ is continuous and strictly monotone on $(\underline{b}, \overline{b}]$. In other words, neither bidder has any gaps in the supports of their distribution, nor any atoms except possibly at $\underline{b}$ or below.*

We strengthen and reinterpret the above lemma with the following extra properties (which are proved in Appendix A).

Lemma 3 *If Assumptions 1, 2 and 3 hold, then any (mixed) Nash Equilibrium F_σ^P of the common-value hybrid auction satisfies the following:*

1. *$\underline{b}^Y = \underline{b}^Z = V(1, 1)$. Thus, the union of supports is the same for both bidders, and the lowest bid in the support of either bidder is the value of the item conditioned on the lowest signal for both bidders.*
2. *At least one of the two bidders deterministically bids $V(1, 1)$ when receiving the lowest signal 1. Both bidders have expected utility 0 when receiving the lowest signal, i.e., $u_1^Y = u_1^Z = 0$.*
3. *The support of the distribution F_σ^P for $\sigma \in \Sigma_P$ is of the form $W_\sigma^P = \langle b_{\sigma-1}^P, b_\sigma^P \rangle$.*[2] *In particular, for a fixed bidder, the supports of his distributions with distinct signals are consecutive intervals, in the region of winning bids.*

Before delving into the equilibrium computation, we examine properties which the cumulative distributions need to satisfy to constitute an equilibrium of the common-value auction. In a common-value setting, the expected value of the item to a bidder depends also on his opponent's signal. Hence, to calculate a bidder's expected utility, we must consider all cases of possible signals for the other bidder. This is captured

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[2] We use $\langle l, r \rangle$ to denote the interval between l and r, which could be open or closed at either end.

by the following formulation of the expected utility:

$$\begin{aligned} u_y^Y(b) = & \sum_{z \in \Sigma_Z} \pi(z|y) \cdot \left(F_z^Z(b) \cdot (V(y,z) - \kappa b) - (1-\kappa) \int_{\underline{b}_z^Z}^{b} \mu dF_z^Z(\mu) \right) \\ = & \sum_{z \in \Sigma_Z} \pi(z|y) \cdot \left(F_z^Z(b) \cdot (V(y,z) - b) + (1-\kappa) \int_{\underline{b}_z^Z}^{b} F_z^Z(\mu) d\mu \right) \end{aligned} \tag{2}$$

where we simply used integration by parts to get the second form of utilities.

Equation (2) can be significantly simplified thanks to Lemmas 2 and 3. Specifically, these lemmas imply that each bid b lies in the support of at most one bidding function of the other bidder. Hence, we can partition the interval $\langle V(1,1), \bar{b} \rangle$ into consecutive intervals $I_{y,z}$, such that each $I_{y,z}$ is the intersection of the supports of a signal y of bidder Y and a signal z of bidder Z. In the interval $I_{y,z}$, we have $F_{y'}^Y(b) = 1$ for all signals $y' < y$, and $F_{y'}^Y(b) = 0$ for all signals $y' > y$. Similarly for bidder Z. Thus for any $b \in I_{y,z}$, we have:

$$\begin{aligned} u_y^Y(b) = & \ \pi(z|y) \cdot \left(F_z^Z(b) \cdot (V(y,z) - b) + (1-\kappa) \int_{\underline{b}_z^Z}^{b} F_z^Z(\mu) d\mu \right) \\ & + \sum_{z' < z} \pi(z'|y) \left(V(y,z') - \kappa b - (1-\kappa) \int_{\underline{b}_{z'}^Z}^{\bar{b}_{z'}^Z} \mu dF_{z'}^Z(\mu) \right). \end{aligned} \tag{3}$$

The cumulative distribution functions constitute an equilibrium if any bid in the support of a bidder's strategy conditional on some signal maximizes his expected utility among all bids. This implies that a bidder must be indifferent among all bids in the support of a strategy, i.e., $u_\sigma^P(b) \equiv u_\sigma^P$ for all bids $b \in W_\sigma^P$. This implies that F_z^Z must satisfy an integral equation of the following form in the interval $I_{y,z}$:

$$F_z^Z(b) \cdot (V(y,z) - b) + (1-\kappa) \int_{\underline{b}_z^Z}^{b} F_z^Z(\mu) d\mu - b\kappa \sum_{z' < z} \frac{\pi(z'|y)}{\pi(z|y)} = \Gamma_{y,z} \tag{4}$$

where $\Gamma_{y,z}$ is some constant associated with the interval $I_{y,z}$. The integral equation 4 yields a solution for the function $\Phi_z^Z(b) = \int_{\underline{b}_z^Z}^{b} F_z^Z(\mu) d\mu$, which can be obtained by standard techniques for solving differential equations. The solution of 4 turns out to be continuous and continuously differentiable at any point except $V(y,z)$. However, by the continuity of $F_z^Z(\mu)$ proved in Lemma 2, Φ_z^Z is continuously differentiable in the interval $I_{y,z}$. Therefore, the differentiability of the solution implies that the interval $I_{y,z}$ must be bounded away from $V(y,z)$. This in turn implies that we can equivalently work with the following differential equation which is well defined in regions that are bounded away from $V(y,z)$:

$$f_z^Z(b)(V(y,z)-b)-\kappa F_z^Z(b)=\kappa\frac{\sum_{z'<z}\pi(z'|y)}{\pi(z|y)}. \tag{5}$$

At the upper bound of $I_{y,z}$, either we know that $F_z^Z(b)=1$, or that F_z^Z is continuous. Either of these gives us a boundary condition for the differential equation. Specifically, let $I_{y,z}=\langle l,r\rangle$, and suppose that the CDFs of the two bidders are already known at the point r. Then, elementary calculus shows that the solution to the differential equation is unique, and for any $b\in I_{y,z}$:

$$\begin{aligned} F_y^Y(b) &= \left(F_y^Y(r)+\frac{\sum_{y'<y}\pi(y'|z)}{\pi(y|z)}\right)\cdot\left(\frac{V(y,z)-r}{V(y,z)-b}\right)^{\kappa}-\frac{\sum_{y'<y}\pi(y'|z)}{\pi(y|z)}, \\ F_z^Z(b) &= \left(F_z^Z(r)+\frac{\sum_{z'<z}\pi(z'|y)}{\pi(z|y)}\right)\cdot\left(\frac{V(y,z)-r}{V(y,z)-b}\right)^{\kappa}-\frac{\sum_{z'<z}\pi(z'|y)}{\pi(z|y)}. \end{aligned} \tag{6}$$

Remark 4 The quantities $\Lambda_Y(y,z)=\frac{\sum_{y'<y}\pi(y'|z)}{\pi(y|z)}$ and $\Lambda_Z(y,z)=\frac{\sum_{z'<z}\pi(z'|y)}{\pi(z|y)}$ play a central role in our equilibrium analysis. They are uniquely determined by the reverse hazard rates of the conditional signal distributions. Recall that the *reverse hazard rate* at x of a distribution with density function $f(x)$ is $R_f(x)=\frac{f(x)}{F(x)}$. For the special case when f is the conditional (discrete) distribution of Y's signal given Z's signal, $\pi(\cdot|z)$, the reverse hazard rate is

$$R_{\pi(\cdot|z)}(y)=\frac{\pi(y|z)}{\sum_{y'\le y}\pi(y'|z)}. \tag{7}$$

Hence, we can write $\Lambda_Y(y,z)=\frac{1}{R_{\pi(\cdot|z)}(y)}-1, \Lambda_Z(y,z)=\frac{1}{R_{\pi(\cdot|y)}(z)}-1$.

The preceding analysis shows that knowing the value of the bidders' CDFs at the upper bound of $I_{y,z}=\langle l,r\rangle$, as well as the upper end r of the interval itself, the CDFs are completely determined for the entire interval. As described above, the value of the CDFs can be inferred by continuity (Lemma 2) from the neighboring interval, or is 1. Our algorithm uses precisely this insight for computing the CDFs interval by interval (though, as we will see, the actual endpoints of intervals can only be determined after the functional forms for the CDFs).

4.1 Existence and Uniqueness of Equilibrium

In this section, we show that there exists a unique mixed Nash Equilibrium of the hybrid common-value auction among two bidders with discrete signals. We give a linear-time algorithm which explicitly computes this unique equilibrium.

4.1.1 Outline

As alluded to in the previous subsection, Lemma 3 guarantees that the interval $\langle V(1,1), \overline{b}\rangle$ can be partitioned into disjoint intervals $I_{y,z}$, such that bids and signals are in a monotone relationship. More formally, the consecutive intervals can be numbered $I_1, \ldots, I_T$ with the following property: $I_t = \langle b_{t-1}, b_t \rangle$ is characterized by two signals $y_t \in \Sigma_Y$ and $z_t \in \Sigma_Z$ with $I_t = I_{y_t,z_t}$. Both y_t and z_t are monotone non-decreasing in t, and $y_t + z_t$ is strictly increasing. Our algorithm will compute the endpoints of the intervals I_t and the CDFs of the two bidders at the upper bound of each interval. Using Equation (6), this completely determines the cumulative distribution functions on the intervals.

As also discussed in the preceding subsection, the continuity of the distribution functions guaranteed by Lemma 2 plays a key role. Specifically, consider two successive intervals t and $t+1$. Let G_t^P be the distribution function of bidder P in the interval I_t, i.e., $G_t^P = F_{\sigma_t}^P$. The following two consequences of Lemmas 2 and 3 will be instrumental in deriving the algorithm:

1. If $\sigma_t < \sigma_{t+1}$, then $G_{t+1}^P(b_t) = 0$ and $G_t^P(b_t) = 1$, by Part 3 of Lemma 3.

2. If $\sigma_t = \sigma_{t+1}$, then $G_t^P(b_t) = G_{t+1}^P(b_t)$ by continuity (Part 3 of Lemma 2).

Starting from the rightmost interval I_T, the algorithm iteratively gives explicit characterizations of G_t^Y and G_t^Z. *The most important part of the analysis is to show that knowing the parameters y_t, z_t, $G_t^Y(b_t)$, $G_t^Z(b_t)$ for the interval I_t is enough to determine these parameters for the neighboring interval I_{t-1}*: more precisely, the algorithm can determine whether $y_t > y_{t-1}$, $z_t > z_{t-1}$, or both, as well as compute $G_{t-1}^Y(b_{t-1})$, $G_{t-1}^Z(b_{t-1})$. From these, in turn, one can explicitly derive G_{t-1}^Y, G_{t-1}^Z over their entire range, and thus continue to the next step. In addition, at each step, the algorithm can determine the lower bound of the interval as a function of the upper bound. The difficulty is that in the first step, the algorithm does not actually know the value of $\overline{b} = b_T$. However, remarkably, all calculations can be performed conditioned on a generic value of $\overline{b}$. The value of $\overline{b}$ is determined when the algorithm reaches an interval I_t with $y_t = 1$ and $z_t = 1$. At that point, the additional equation that $u_1^Y = u_1^Z = 0$ provides the extra equation to determine $\overline{b}$.

Consider an arbitrary interval I_t, for which $y_t, z_t, G_t^Y(b_t), G_t^Z(b_t)$ are known, and $y_t > 1$ or $z_t > 1$. The first goal is to determine the signals corresponding to the interval I_{t-1}, i.e., whether $y_t > y_{t-1}$, $z_t > z_{t-1}$, or both. (By the definition of the intervals, at least one of these inequalities must hold.) The lower endpoint b_{t-1} of I_t is determined by the fact that $\min(G_t^Y(b_{t-1}), G_t^Z(b_{t-1})) = 0$ (since no bidder can have an atom in the winning region of bids). Let b, b' solve $G_t^Y(b) = 0$ and $G_t^Z(b') = 0$, respectively. Then, $b_{t-1} = \max(b, b')$. If $b > b'$ then $y_{t-1} = y_t - 1, z_{t-1} = z_t$, whereas $y_{t-1} = y_t, z_{t-1} = z_t - 1$ if $b > b'$. (If $b = b'$, then

$y_{t-1} = y_t - 1, z_{t-1} = z_t - 1.$)

Equation (6) provides explicit forms for G_t^Y and G_t^Z. Using these forms to solve $G_t^Y(b) = 0$ and $G_t^Z(b') = 0$ gives us that

$$\left(\frac{V(y_t, z_t) - b_t}{V(y_t, z_t) - b}\right)^{\kappa} = \frac{1}{1 + \frac{G_t^Y(b_t)}{\Lambda_Y(y,z)}} \qquad \left(\frac{V(y_t, z_t) - b_t}{V(y_t, z_t) - b'}\right)^{\kappa} = \frac{1}{1 + \frac{G_t^Z(b_t)}{\Lambda_Z(y,z)}}.$$

While computing b or b' requires knowing b_t (and thus implicitly $\bar{b}$), determining whether $b > b'$ or $b < b'$ does *not* require knowledge of b_t, only of $y_t, z_t, G_t^Y(b_t), G_t^Z(b_t)$. This is the key insight that allows the algorithm to proceed correctly through all intervals. Since both functions $G_t^Y(b), G_t^Z(b)$ are monotone increasing, we know that $b_t \leq V(y, z)$. Thus the comparison $b \geq b'$ is equivalent to:

$$\frac{G_t^Z(b_t)}{\Lambda_Z(y_t, z_t)} \geq \frac{G_t^Y(b_t)}{\Lambda_Y(y_t, z_t)}, \tag{8}$$

which is a comparison that depends only on the quantities y_t, z_t, $G_t^Y(b_t)$, $G_t^Z(b_t)$ known to the algorithm from the previous iteration.

4.1.2 The "Main Recursion"

The algorithm therefore distinguishes three cases:

1. If $\frac{G_t^Z(b_t)}{\Lambda_Z(y_t,z_t)} < \frac{G_t^Y(b_t)}{\Lambda_Y(y_t,z_t)}$, then the parameters of the next lower interval I_{t-1} are:

$$\begin{aligned} y_{t-1} &= y_t \\ z_{t-1} &= z_t - 1 \\ b_{t-1} &= V(y_t, z_t) - (V(y_t, z_t) - b_t)\left(1 + \frac{G_t^Z(b_t)}{\Lambda_Z(y_t, z_t)}\right)^{1/\kappa} \\ G_{t-1}^Y(b_{t-1}) &= \frac{G_t^Y(b_t)\Lambda_Z(y_t, z_t) - G_t^Z(b_t)\Lambda_Y(y_t, z_t)}{G_t^Z(b_t) + \Lambda_Z(y_t, z_t)} \\ G_{t-1}^Z(b_{t-1}) &= 1. \end{aligned}$$

2. If $\frac{G_t^Z(b_t)}{\Lambda_Z(y_t,z_t)} > \frac{G_t^Y(b_t)}{\Lambda_Y(y_t,z_t)}$, then the parameters for the next lower interval I_{t-1} are

$$\begin{aligned}
y_{t-1} &= y_t - 1 \\
z_{t-1} &= z_t \\
b_{t-1} &= V(y_t, z_t) - (V(y_t, z_t) - b_t)\left(1 + \frac{G_t^Y(b_t)}{\Lambda_Y(y_t, z_t)}\right)^{1/\kappa} \\
G_{t-1}^Y(b_{t-1}) &= 1 \\
G_{t-1}^Z(b_{t-1}) &= \frac{G_t^Z(b_t)\Lambda_Y(y_t, z_t) - G_t^Y(b_t)\Lambda_Z(y_t, z_t)}{G_t^Y(b_t) + \Lambda_Y(y_t, z_t)}.
\end{aligned}$$

3. If $\frac{G_t^Z(b_t)}{\Lambda_Z(y_t,z_t)} = \frac{G_t^Y(b_t)}{\Lambda_Y(y_t,z_t)}$, then the values for the next lower interval I_{t-1} are

$$\begin{aligned}
y_{t-1} &= y_t - 1 \\
z_{t-1} &= z_t - 1 \\
b_{t-1} &= V(y_t, z_t) - (V(y_t, z_t) - b_t)\left(1 + \frac{G_t^Y(b_t)}{\Lambda_Y(y_t, z_t)}\right)^{1/\kappa} \\
G_{t-1}^Y(b_{t-1}) &= 1 \\
G_{t-1}^Z(b_{t-1}) &= 1.
\end{aligned}$$

4.1.3 The Lower End

The cases $y_t = 1$ or $z_t = 1$ require some special treatment. The recurrence terminates when both $y_t = 1$ and $z_t = 1$. However, we must ensure that the recurrence never calls for the case of decreasing either $y_t = 1$ or $z_t = 1$ to zero.

Assume without loss of generality that $y_t > 1, z_t = 1$. Then, the differential equation defining $G_t^Z(b)$ is the homogeneous differential equation corresponding to Equation (5) in which the right-hand side is 0. This yields a unique solution for all $b \in I_t$:

$$G_t^Z(b) = G_t^Z(b_t)\left(\frac{V(y_t, 1) - b_t}{V(y_t, 1) - b}\right)^{\kappa}.$$

We will argue that $V(y_t, z_t) > b_t$, which implies that $G_t^Z(b) > 0$ for the entire interval; in particular, the recurrence can never reach the (invalid) case that $z_t = 0$.

We show the inequality $V(y_t, z_t) > b_t$. By the analysis of the algorithm, the upper bound of interval t is given by the following formula for $P \in \{Y, Z\}$:

$$b_t = V(y_t, z_t) - (V(y_t, z_t) - b_{t-1}) \left(1 + \frac{G_t^P(b_t)}{\Lambda_P(y_t, z_t)}\right)^{-1/\kappa}. \tag{9}$$

Since $V(y, z)$ is strictly increasing in both arguments and since $b_0 = V(1, 1)$, the above equality shows inductively that each upper bound b_t must be strictly smaller than $V(y_t, z_t)$.

Finally, when the algorithm reaches the point $y_t = z_t = 1$, the lower bound of the interval I_{t+1} is known to be $V(1, 1)$. This value can be substituted iteratively into the Equations (9) computed during the first pass. This completes the computation of the explicit form of the unique equilibrium.

4.1.4 Verification

To complete the analysis, it remains to show that the unique solution computed by the algorithm actually constitutes an equilibrium of the hybrid auction. The essence of the proof is the following lemma, which establishes that the assumption of affiliation of the bidders' signals implies a reverse hazard rate dominance relation on the conditional distributions of opponent's signals.

Lemma 5 *If the signals of the bidders are* affiliated*, then the distributions $\pi(\cdot|\sigma)$ are completely ordered according to Likelihood Ratio Dominance and hence according to Reverse Hazard Rate Dominance, as a function of the signal σ.*

Proof. By the definition of affiliation, for any signals $y' \leq y \in \Sigma_Y$ and $z' \leq z \in \Sigma_Z$,

$$\pi(y, z)\pi(y', z') \geq \pi(y', z)\pi(y, z').$$

Dividing this equation by $\pi_Y(y)\pi_Y(y')$, we obtain that $\pi(z|y)\pi(z'|y') \geq \pi(z|y')\pi(z'|y)$, which implies that

$$\frac{\pi(z|y)}{\pi(z|y')} \geq \frac{\pi(z'|y)}{\pi(z'|y')}.$$

This inequality captures that the distribution $\pi(\cdot|y)$ stochastically dominates the distribution $\pi(\cdot|y')$ according to the likelihood-ratio order, which is known to also imply reverse hazard rate dominance. ■

Remark 6 The equilibrium computed by the algorithm remains an equilibrium even if we just assume Reverse Hazard Rate Dominance of the distributions and not affiliation of the signals. Hence, the equilibrium characterization is valid for a more general class of signal distributions.

Theorem 7 *If signals are* informative *and the conditional distributions $\pi(\cdot|\sigma)$ are stochastically ordered according to the Reverse Hazard Rate order, with respect to σ, then the unique solution defined by our algorithm is a mixed Nash Equilibrium of the hybrid auction.*

Proof. The continuity of the CDFs implies that the utility $u^P_\sigma(b)$ of a bidder conditional on a signal σ, as defined in Equation (3), is a continuous function of b on the whole positive real line. Because the CDFs are explicitly determined to solve the given differential equations, the utility function $u^P_\sigma(b)$ must be constant over the support of F^P_σ. It therefore remains to be shown is that the utility of P receiving the signal σ can be no larger outside the support.

To show this, we prove that the derivative of $u^P_\sigma(b)$ is non-positive within any interval I_t associated with a signal $\sigma' > \sigma$ and non-negative within any interval associated with a signal $\sigma' < \sigma$. Combined with the continuity of $u^P_\sigma(b)$, this implies that the utility is maximized in the support of F^P_σ.

We will prove the above for a signal y of bidder Y. Consider an interval $I_{y',z}$ associated with a signal $y' > y$. By the definition of the Main Recursion, the distribution F^Z_z in that interval satisfies the differential equation (5):

$$f^Z_z(b) \cdot (V(y', z) - b) - \kappa F^Z_z(b) = \kappa \Lambda_Z(y', z).$$

The derivative of $u^Y_y(b)$ in that interval is

$$\begin{aligned}\left(u^Y_y(b)\right)' &= \pi(z|y) \cdot \left(f^Z_z(b) \cdot (V(y, z) - b) - \kappa F^Z_z(b)\right) - \kappa \sum_{z'<z} \pi(z'|y) \\ &= \pi(z|y) \cdot \left(f^Z_z(b) \cdot (V(y, z) - b) - \kappa F^Z_z(b) - \kappa \Lambda_Z(y, z)\right).\end{aligned}$$

The informativeness of the signals implies that $V(y, z) < V(y', z)$, and the reverse hazard rate dominance of the conditional distributions implies that $\Lambda_Z(y, z) \geq \Lambda_Z(y', z)$. Thus,

$$\left(u^Y_y(b)\right)' \leq \pi(z|y) \cdot \left(f^Z_z(b) \cdot (V(y', z) - b) - \kappa F^Z_z(b) - \kappa \Lambda_Z(y', z)\right) \; = \; 0.$$

The proof that the utility is non-decreasing for intervals associated with smaller signals is analogous. ■

4.2 Symmetric Equilibrium

In this section, we characterize a necessary and sufficient condition for the unique Nash Equilibrium to be symmetric. Our condition significantly generalizes a sufficient condition established by Hausch [16].

First, for a symmetric equilibrium, it is clearly necessary for the bidders to have identical signal spaces $\Sigma_Y = \Sigma_Z = \{1, \ldots, k\}$. Furthermore, the supports of F_σ^Y and F_σ^Z must be the same for each signal σ, implying that it is necessary for the algorithm to always choose the third case of the recursion. This is the case if and only if $\Lambda_Y(\sigma, \sigma) = \Lambda_Z(\sigma, \sigma)$ for all signals σ. Hence, we have derived the following corollary:

Corollary 8 *The unique equilibrium is symmetric if and only if the signal spaces of both bidders have the same size k, and $\Lambda_Y(\sigma, \sigma) = \Lambda_Z(\sigma, \sigma)$ for each $\sigma \in \{1, \ldots, k\}$.*

Written more elaborately, the condition of Corollary 8 is that for all signals $\sigma \in \{1, \ldots, k\}$:

$$\frac{\mathbb{P}\left(S_Y = \sigma | S_Z = \sigma\right)}{\sum_{\sigma' \leq \sigma} \mathbb{P}\left(S_Y = \sigma' | S_Z = \sigma\right)} = \frac{\mathbb{P}\left(S_Z = \sigma | S_Y = \sigma\right)}{\sum_{\sigma' \leq \sigma} \mathbb{P}\left(S_Z = \sigma' | S_Y = \sigma\right)}.$$

Hence, the condition simply states that the reverse hazard rate of the opponent's distribution at the equivalent signal is the same for both bidders.

Hausch [16] showed that the first-price common-value auction has a symmetric equilibrium under the following stronger condition: $\mathbb{P}\left(S_Y = \sigma' | S_Z = \sigma\right) = \mathbb{P}\left(S_Z = \sigma' | S_Y = \sigma\right)$ for any pair of signals σ, σ'. This condition trivially implies the condition of Corollary 8; hence, our result generalizes the sufficient condition of Hausch.

Under the condition of Corollary 8, the calculations of the (common) CDFs can be simplified. The CDFs are now of the form

$$F_\sigma^P(b) = \left(1 + \Lambda_P(\sigma, \sigma)\right) \cdot \left(\frac{V(\sigma, \sigma) - r}{V(\sigma, \sigma) - b}\right)^\kappa - \Lambda_P(\sigma, \sigma). \tag{10}$$

The recurrence for the intervals' endpoints is also greatly simplified:

$$b_\sigma = V(\sigma, \sigma) - \left(V(\sigma, \sigma) - b_{\sigma-1}\right)\left(1 + \frac{1}{\Lambda_P(\sigma, \sigma)}\right)^{-\frac{1}{\kappa}} = V(\sigma, \sigma) - \left(V(\sigma, \sigma) - b_{\sigma-1}\right)\left(\frac{\sum_{\sigma' < \sigma} \pi(\sigma' | \sigma)}{\sum_{\sigma' \leq \sigma} \pi(\sigma' | \sigma)}\right)^{\frac{1}{\kappa}},$$

and the base case is $b_0 = V(1, 1)$.

4.3 Invariance of the Tying Function

In this section, we introduce a tying function for discrete distributions and show the interesting property that the tying function is the same for all hybrid auctions, independent of κ. The insight will be helpful in our revenue ranking results in Section 5. An analogue of this result for the continuous setting was shown by Parreiras [32].

To define the tying function, we consider a two-dimensional extension of a bidder's signal used previously by Milgrom et al. [29, 12]. We can think of the mixed equilibrium as follows: suppose that together with his true signal σ, each bidder receives a second signal ρ distributed uniformly at random in $[0,1]$; we call ρ the *quantile* of bidder P and the pair (σ,ρ) the *extended signal* of bidder P. Now we can describe the mixed strategy of each bidder as a deterministic strategy $b_P(\sigma,\rho)$ based on the extended signal: $b_P(\sigma,\rho) = (F^P_\sigma)^{-1}(\rho)$, i.e., $b_P(\sigma,\rho)$ is the bid b such that bidder P, when receiving signal σ, bids below b with probability ρ.

Given this formulation, we can now define the *tying function* between the two bidders in the discrete information setting. $Q(y,\rho_Y) = (z,\rho_Z)$ is defined as the solution to the equation $b_Y(y,\rho_Y) = b_Z(Q(y,\rho_Y))$, i.e., $Q(y,\rho_Y)$ is the extended signal for bidder Z under which Z submits the same bid as bidder Y submits with the extended signal (y,ρ_Y).

Lemma 9 *The tying function of any hybrid auction at its unique equilibrium is independent of the probability κ that the winner pays his bid.*

Proof. First, we note that from the equilibrium analysis, the interval structure of the equilibrium is independent of κ, since inductively, the comparisons of $\frac{G^Z_t(b_t)}{\Lambda_Z(y_t,z_t)}$ vs. $\frac{G^Y_t(b_t)}{\Lambda_Y(y_t,z_t)}$ and the assignments of new values for $G^P_{t-1}(b_{t-1})$ do not depend on the value of κ. Hence, the global interval structure (i.e., the y_t and z_t) and the CDFs at the interval endpoints do not depend on κ.

Now consider a signal y of bidder Y and a quantile ρ_Y. The quantile ρ_Y determines a bid for Y, which will fall into some interval I_t, independently of the parameter κ. The interval I_t has a corresponding signal z_t of bidder Z. I_t is determined by the condition $F^Y_\sigma(b_{t-1}) \le \rho_Y \le F^Y_\sigma(b_t)$; by the previous paragraph, this condition does not depend on κ.

In the interval I_t, the CDFs of the two bidders are defined by the Equations (6). For a given κ, let $b_Y(y,\rho_Y)$ be the bid corresponding to an extended signal of bidder Y. This bid must solve

$$\rho_Y = \left(F^Y_y(b_t) + \Lambda_Y(y,z_t)\right) \cdot \left(\frac{V(y,z_t) - b_t}{V(y,z_t) - b_Y(y,\rho_Y)}\right)^\kappa - \Lambda_Y(y,z_t).$$

Solving for $\left(\frac{V(y,z_t)-b_t}{V(y,z_t)-b_Y(y,\rho_Y)}\right)^\kappa$ and substituting into Equation (6) for bidder Z's CDF, we get that

$$F^Z_{z_t}(b_Y(y,\rho_Y)) = \left(F^Z_{z_t}(b_t) + \Lambda_Z(y,z_t)\right) \cdot \frac{\rho_Y + \Lambda_Y(y,z_t)}{F^Y_y(b_t) + \Lambda_Y(y,z_t)} - \Lambda_Z(y,z_t).$$

Hence, the value of the tying function at (y,ρ_Y) is

$$Q(y,\rho_Y) = \left(z_t\ ,\ \left(F^Z_{z_t}(b_t) + \Lambda_Z(y,z_t)\right) \cdot \frac{\rho_Y + \Lambda_Y(y,z_t)}{F^Y_y(b_t) + \Lambda_Y(y,z_t)} - \Lambda_Z(y,z_t)\right),$$

which is independent of κ. ■

5 Complete Revenue Ranking

In this section, we study the effect of κ — the interpolation parameter between first and second-price auctions — on the bidders' utilities and the auctioneer's revenue R. We show the following monotonicity properties:

Theorem 10 (Revenue Ranking) *If Assumptions 1, 2 and 3 are satisfied, then the following hold:*

1. *Each bidder's ex-interim expected utility u^P_σ is monotone non-decreasing in κ.*
2. *The auctioneer's expected revenue $\mathrm{E}[R]$ is a monotone non-increasing function of κ.*

Most of the proof will be concerned with showing the first property. The second is a simple corollary of the first. In order to analyze a bidder's expected utility, we will primarily evaluate it at the specific point $\bar{b}^P$, i.e., we use the fact that $u^P_\sigma = u^P_\sigma(\bar{b}^P_\sigma)$. In order to carry out this analysis, we first need to understand how the $\bar{b}^P_\sigma$ depend on κ. This is captured by the following technical lemma.

Lemma 11 *The upper bounds $\bar{b}^P_\sigma$ of the equilibrium distributions F^P_σ are non-increasing in κ.*

Proof. By Lemma 9, the global interval structure (i.e., y_t and z_t) and the CDFs at the interval endpoints do not depend on κ. The only parameters of the distribution that actually depend on κ are the values of the intervals' upper and lower bounds themselves.

We now prove the lemma by induction on the interval t. The lower bound of the first interval is $b_0 = V(1,1)$, irrespective of κ. By the algorithm's recursive step, the upper bound b_t can be expressed in terms of b_{t-1} as

$$b_t = V(y_t,z_t) - (V(y_t,z_t) - b_{t-1})\left(1 + \frac{G^P_t(b_t)}{\Lambda_P(y_t,z_t)}\right)^{-1/\kappa}$$

for some bidder $P \in \{Y, Z\}$. By induction, b_{t-1} is non-increasing in κ; hence, $(V(y_t, z_t) - b_t)$ is non-decreasing in κ. As explained above, $1 + \frac{G_t^P(b_t)}{\Lambda_P(y_t, z_t)}$ is independent of κ, so $\left(1 + \frac{G_t^P(b_t)}{\Lambda_P(y_t, z_t)}\right)^{-1/\kappa}$ is non-decreasing in κ. Overall, b_t must therefore be non-increasing in κ. ■

Proof of Theorem 10. We begin by showing that the revenue monotonicity (the second part of the theorem) is a direct consequence of the bidders' utility monotonicity (first part). The ex-ante expected utility of bidder P is $\sum_{\sigma \in \Sigma_P} \pi_P(\sigma) u_\sigma^P$. Because each of the terms u_σ^P is monotone non-decreasing in κ, and the $\pi_P(\sigma)$ are independent of κ, each bidder's utility is non-decreasing in κ. The auctioneer's expected revenue $\mathrm{E}[R]$ is the expected value $\mathrm{E}[v]$ of the item (independent of κ), minus the sum of the two bidders' utilities. Hence, it is monotone non-increasing in κ.

The remainder of the proof will focus on the first part of the theorem. Focus on bidder Y with signal y. By Equation (2), Y's utility is

$$u_y^Y(b) = \sum_{z \in \Sigma_Z} \pi(z|y) \cdot \left(F_z^Z(b) \cdot (V(y,z) - b) + (1-\kappa) \int_{\underline{b}_z^Z}^{b} F_z^Z(\mu) d\mu \right).$$

Because Y's utility is constant over the support of F_y^Y, we can evaluate it at the highest point in the support of F_y^Y; let τ be the index of the interval such that this point is b_τ. (Formally, τ is maximal such that $y_\tau \leq y$.) Then, we can rewrite Y's utility as

$$u_y^Y = \sum_{z \in \Sigma_Z} \pi(z|y) \cdot \left(F_z^Z(b_\tau) \cdot (V(y,z) - b_\tau) + (1-\kappa) \int_{\underline{b}_z^Z}^{b_\tau} F_z^Z(\mu) d\mu \right).$$

Lemma 9 guarantees that the interval structure itself and the CDFs at the upper bounds, $F_z^Z(b_t)$, are independent of κ, while Lemma 11 guarantees that the upper bounds b_t are non-increasing in κ. Therefore,

$$\sum_{z \in \Sigma_Z} \pi(z|y) \cdot F_z^Z(b_\tau) \cdot (V(y,z) - b_\tau)$$

is non-decreasing in κ. It therefore suffices to show that

$$B = \sum_{z \in \Sigma_Z} \pi(z|y) \cdot (1-\kappa) \int_{\underline{b}_z^Z}^{b_\tau} F_z^Z(\mu) d\mu$$

is also non-decreasing in κ. We will prove the stronger statement that for any interval $t \in \{1, \ldots, \tau\}$:

$$Q_t = \sum_{z \in \Sigma_Z} \pi(z|y) \cdot (1-\kappa) \int_{b_{t-1}}^{b_t} F_z^Z(\mu) d\mu \tag{11}$$

is non-decreasing in κ. As $B = \sum_{t \leq \tau} Q_t$, the bounds on Q_t imply the desired bound on B. Notice that we are crucially exploiting here that the interval structure (i.e., the values y_t, z_t) is independent of κ.

Consider the interval I_t, corresponding to the supports of distributions for signals y_t, z_t. For signals $z < z_t$, we have that $F_z^Z(b) = 1$, while for $z > z_t$, $F_z^Z(b) = 0$. For the signal $z = z_t$, Equation (6) gives us the following form:

$$F_{z_t}^Z(b) = \left(F_{z_t}^Z(b_t) + \Lambda_Z(y_t, z_t)\right) \cdot \left(\frac{V(y_t, z_t) - b_t}{V(y_t, z_t) - b}\right)^{\kappa} - \Lambda_Z(y_t, z_t).$$

Integrating this equality over the interval $[b_{t-1}, b_t]$, we get:

$$\int_{b_{t-1}}^{b_t} F_{z_t}^Z(\mu) d\mu = \left[\left(F_{z_t}^Z(b_t) + \Lambda_Z(y_t, z_t)\right) \cdot (V(y_t, z_t) - b_t)^{\kappa} \cdot \frac{(V(y_t, z_t) - \mu)^{1-\kappa}}{1-\kappa} - \Lambda_Z(y_t, z_t)\mu\right]_{b_{t-1}}^{b_t} =$$

$$\left(F_{z_t}^Z(b_t) + \Lambda_Z(y_t, z_t)\right) \cdot (V(y_t, z_t) - b_t)^{\kappa} \cdot \frac{(V(y_t, z_t) - b_t)^{1-\kappa} - (V(y_t, z_t) - b_{t-1})^{1-\kappa}}{1-\kappa} - \Lambda_Z(y_t, z_t)(b_t - b_{t-1}).$$

By Equation (9), b_t and b_{t-1} relate as follows:

$$b_t = V(y_t, z_t) - (V(y_t, z_t) - b_{t-1}) \cdot \left(1 + \frac{F_\sigma^P(b_t)}{\Lambda_P(y_t, z_t)}\right)^{-1/\kappa}$$

for some bidder P. For notational convenience, we write $a_t = 1 + \frac{F_\sigma^P(b_t)}{\Lambda_P(y_t, z_t)} \geq 1$; as argued in the proof of Lemma 11, a_t is independent of κ. From the equation relating b_t with b_{t-1}, we can derive the following useful identities:

$$\begin{aligned} V(y_t, z_t) - b_{t-1} &= (V(y_t, z_t) - b_t)\, a_t^{1/\kappa}, \\ b_t - b_{t-1} &= (V(y_t, z_t) - b_{t-1})\,(1 - a_t^{-1/\kappa}) = (V(y_t, z_t) - b_t)\,(a_t^{1/\kappa} - 1). \end{aligned}$$

Substituting these identities into the formula for $\int_{b_{t-1}}^{b_t} F_z^Z(\mu) d\mu$ leads to the following simplification:

$$\int_{b_{t-1}}^{b_t} F_{z_t}^Z(\mu) d\mu = (V(y_t, z_t) - b_t) \cdot \left(\frac{F_{z_t}^Z(b_t) + \Lambda_Z(y_t, z_t)}{1-\kappa} \cdot \left(1 - a_t^{(1-\kappa)/\kappa}\right) - \Lambda_Z(y_t, z_t)(a_t^{1/\kappa} - 1)\right).$$

For signals $z > z_t$, the integral is 0, and for signals $z < z_t$, the integral is

$$\int_{b_{t-1}}^{b_t} F_z^Z(\mu)d\mu \;=\; b_t - b_{t-1} \;=\; (V(y_t, z_t) - b_t)\,(a_t^{1/\kappa} - 1).$$

Combining all of these cases, we derive that

$$Q_t = (V(y_t, z_t) - b_t) \cdot$$
$$\left(\pi(z_t|y) \cdot \left(F_{z_t}^Z(b_t) + \Lambda_Z(y_t, z_t)\right) \cdot \left(1 - a_t^{(1-\kappa)/\kappa}\right) + (1-\kappa)(1 - a_t^{1/\kappa}) \cdot \left(\pi(z_t|y)\Lambda_Z(y_t, z_t) - \sum_{z<z_t} \pi(z|y)\right)\right).$$

By Lemma 11, $(V(y_t, z_t) - b_t)$ is an non-decreasing in κ, and we will show that the second term is non-decreasing in κ as well. First, notice that $a_t, \pi(z_t|y)$, $G_t^Z(b_t)$ and $\Lambda_Z(y_t, z_t)$ are all independent of κ. As $a_t \geq 1$, $1 - a_t^{(1-\kappa)/\kappa} = 1 - a_t^{1/\kappa - 1}$ is non-decreasing in κ, and so is the entire first term.

For the second term, because we are considering an interval $t < \tau$, the ordering of intervals implies that $y_t \leq y$. Because the signals are affiliated by assumption, $\Lambda_Z(y_t, z_t) \geq \Lambda_Z(y, z_t)$ which implies that

$$\pi(z_t|y)\Lambda_Z(y_t, z_t) \geq \sum_{z<z_t} \pi(z|y).$$

A simple derivative test shows that $(1-\kappa)(1 - a_t^{1/\kappa})$ is non-decreasing in κ. Thus, the second term is also increasing in κ, completing the proof. ■

6 Equilibrium Selection for Second-Price Auctions

It is well known that the second-price common-value auction generally has many equilibria. Thus, for any type of analysis of second-price auctions, it is necessary to select the equilibria to study. Indeed, several past works (see, e.g., [1, 24, 10, 6, 21, 32]) have proposed different ways of selecting equilibria of the second-price auction. One natural equilibrium procedure is to study the limit of the hybrid auction as $\kappa \to 0$. Our explicit characterization makes the computation of this lemma straightforward.

When we consider the limit of Equation (9) as $\kappa \to 0$, we obtain that $b_t = V(y_t, z_t)$, and as before, $b_0 = V(1, 1)$. Combining this simplification with the recursive structure of the equilibrium computation algorithm, the interval structure and density functions at the upper bounds in the limit $\kappa \to 0$ are recursively defined as follows:

$$b_{t-1} = \begin{cases} V(y_t, z_t - 1) & \text{if } \frac{G_t^Z(b_t)}{\Lambda_Z(y_t,z_t)} < \frac{G_t^Y(b_t)}{\Lambda_Y(y_t,z_t)} \\ V(y_t - 1, z_t) & \text{if } \frac{G_t^Z(b_t)}{\Lambda_Z(y_t,z_t)} > \frac{G_t^Y(b_t)}{\Lambda_Y(y_t,z_t)} \\ V(y_t - 1, z_t - 1) & \text{if } \frac{G_t^Z(b_t)}{\Lambda_Z(y_t,z_t)} = \frac{G_t^Y(b_t)}{\Lambda_Y(y_t,z_t)}, \end{cases} \tag{12}$$

$$G_{t-1}^Y(b_{t-1}) = \begin{cases} 1 & \text{if } \frac{G_t^Z(b_t)}{\Lambda_Z(y_t,z_t)} \geq \frac{G_t^Y(b_t)}{\Lambda_Y(y_t,z_t)} \\ \frac{G_t^Y(b_t)\Lambda_Z(y_t,z_t) - G_t^Z(b_t)\Lambda_Y(y_t,z_t)}{G_t^Z(b_t) + \Lambda_Z(y_t,z_t)} & \text{otherwise,} \end{cases} \tag{13}$$

$$G_{t-1}^Z(b_{t-1}) = \begin{cases} 1 & \text{if } \frac{G_t^Y(b_t)}{\Lambda_Y(y_t,z_t)} \geq \frac{G_t^Z(b_t)}{\Lambda_Z(y_t,z_t)} \\ \frac{G_t^Z(b_t)\Lambda_Y(y_t,z_t) - G_t^Y(b_t)\Lambda_Z(y_t,z_t)}{G_t^Y(b_t) + \Lambda_Y(y_t,z_t)} & \text{otherwise.} \end{cases} \tag{14}$$

By Equation (6) with $\kappa \to 0$, we get that $G_t^P(b) = G_t^P(b_t)$ for all $b \in I_t$, i.e., the CDFs of both bidders are constant, which means that they jump at interval boundaries. Thus, the bidders randomize only over discrete sets, and the respective probabilities of bidding b_t can be simply computed as the difference $G_t^P(b_t) - G_{t-1}^P(b_{t-1})$ or $G_t^P(b_t)$, depending on whether $\sigma_{t-1} = \sigma_t$ or $\sigma_{t-1} = \sigma_t - 1$. Thus, we can explicitly characterize the structure of the second-price auction equilibrium selected as the limit of hybrid equilibria.

6.1 Symmetric Equilibrium

Under the condition of Corollary 8, which implies that the equilibrium is symmetric, the following symmetric equilibrium of the second price auction is selected: *conditional on receiving a signal σ, each bidder deterministically bids $V(\sigma, \sigma)$.* This equilibrium is the same as the one presented by Milgrom and Weber [28] for the setting of symmetric and continuous signals. Hence, our analysis generalizes this equilibrium to asymmetric signal structures satisfying the condition of Corollary 8, and to discrete signal distributions. The expected utility for the bidders at this symmetric equilibrium is

$$u^Y = \sum_{\sigma \in \Sigma} \sum_{\sigma' < \sigma} \pi(\sigma, \sigma') \cdot (V(\sigma, \sigma') - V(\sigma', \sigma')) \; > \; 0, \tag{15}$$

$$u^Z = \sum_{\sigma \in \Sigma} \sum_{\sigma' < \sigma} \pi(\sigma', \sigma) \cdot (V(\sigma', \sigma) - V(\sigma', \sigma')) \; > \; 0. \tag{16}$$

6.2 Uninformed Bidders and the Robustness of Free-Riding

To gain more intuition about the nature of the selected equilibrium, and to examine the robustness of the observed free-riding behavior, we consider the limit equilibrium selected when bidder Z is completely uninformed. More formally, Z deterministically receives the signal 1, while Y receives a signal from some discrete set Σ_Y according to some distribution.

In this case, there will be exactly k_Y intervals, each characterized by a signal y of bidder Y, with interval 1 being the single point $V_Y(1)$. The upper bounds of each of these intervals are $V_Y(y)$.

In the special case we consider, we always have that $y_{t-1} = y_t - 1$, and thus $G_t^Y(b_t) = 1$ as well. Since we have that $y_t = t$, we will simply identify the interval t with bidder Y's signal y. Since the second case of Equation (14) always applies, the calculation of the recurrence for the CDF of bidder Z simplifies to the following:

$$G_{y-1}^Z(b_{y-1}) \;=\; G_y^Z(b_y)\frac{\Lambda_Y(y,1)}{1+\Lambda_Y(y,1)} \;=\; G_y^Z(b_y)\frac{\mathbb{P}(S_Y < y)}{\mathbb{P}(S_Y < y+1)}.$$

Using $G_{k_Y}^Z(b_{k_Y}) = 1$ as a base case, a straightforward induction now shows that $G_y^Z(b_y) = \mathbb{P}(S_Y \le y)$. In other words, the limit strategy of the uninformed bidder is to bid $V_Y(y)$ with probability $\pi_Y(y)$. On the other hand, the informed bidder bids $V_Y(y)$ deterministically when receiving signal y, which happens with probability $\pi_Y(y)$. Thus, the two bidders have the same ex ante distribution of bids.

Notice that the limit equilibrium above involves extensive free-riding by the uninformed bidder. Thanks to Y's bidding strategy, Z ensures that his payment conditional on winning is the expected value of the item; in other words, he takes advantage of Y's information.

6.2.1 Revenue Equivalence

Interestingly, the equilibrium whose selection we just described is revenue- and utility-equivalent to the equilibrium of the first-price auction (and thus, by Theorem 10, to the equilibrium of the hybrid auction for any κ).

By examining the recurrence for the equilibrium, or by using the characterization of Engelbrecht-Wiggans et al. [12] (who study first-price auctions with only one informed bidder), we obtain that in a first-price auction with one uninformed bidder, the upper bound of interval $y \in [1, k_Y]$ is $\mathrm{E}[v \mid S_Y \le y]$. Hence, the expected utility of bidder Y conditional on signal y in the first-price auction is

$$u_y^Y \;=\; G_y^Z(b_y)\,(V_Y(y) - b_y) \;=\; \mathbb{P}(S_Y \le y)\cdot(V_Y(y) - \mathrm{E}[v \mid S_Y \le y]).$$

On the other hand, in the limit equilibrium of the second-price auction as $\kappa \to 0$, the expected utility of bidder Y conditional on signal y is

$$\begin{aligned} u_y^Y &= \sum_{y' \leq y} \mathbb{P}\left(S_Y = y'\right) \cdot \left(V_Y(y) - V_Y(y')\right) \\ &= V_Y(y) \cdot \mathbb{P}\left(S_Y \leq y\right) - \sum_{y' \leq y} \mathbb{P}\left(S_Y = y'\right) \cdot V_Y(y') \\ &= \mathbb{P}\left(S_Y \leq y\right)\left(V_Y(y) - \mathrm{E}\left[v \mid S_Y \leq y\right]\right). \end{aligned}$$

Thus, bidder Y's expected utility conditional on any signal is the same in the first- and the second-price auction (for the selected equilibrium). Bidder Z's utility is 0 in both auctions. Therefore, the expected revenue of the two auctions is also the same.

7 Fully Binary Instances of First-Price Auctions

In this section, we illustrate the algorithm from Section 4 by focusing on the special case of a first-price auction in which both the item's value and the bidders' signals are binary. The analysis also forms the basis for an exploration of the failure of the Linkage Principle and surprising non-monotonicity properties in Sections 8 and 9.

Definition 12 *An instance of the hybrid auction is* fully binary *if v takes values only in $\{0, 1\}$, and $\Sigma_Y = \Sigma_Z = \{1, 2\}$.*

In that case, we write $\alpha = \mathbb{P}(v = 1)$ for the probability that the item is valuable, and q_2^P, p_2^P for the probability that bidder $P \in \{Y, Z\}$ sees the signal 2 conditioned on the item's value being 0 and 1, respectively. (Thus, $q_1^P = 1 - q_2^P, p_1^P = 1 - p_2^P$.)

For some of our subsequent analysis, it is also helpful to restrict the instances further, requiring the types of error of a bidder's signal to be "symmetric," in the following sense:

Definition 13 *In the fully binary setting, a bidder P has* symmetric error *if $p_2^P = q_1^P$. In this case, we write $p_P := p_2^P = q_1^P$.*

In words, if a bidder's signal has symmetric error, then conditioned on any actual value of the item, the bidder's signal reveals the true value with probability p_P, and the opposite of the correct value with probability $1 - p_P$. In this sense, p_P is a natural — or rather, the unique — measure of the quality of a bidder's signal.

Banerjee [4] analyzes equilibria of first-price auctions in a setting resembling the fully binary one; however, the analysis is simplified significantly by having values drawn from $\{-c_1, c_2\}$ with $c_1 > 0$ large enough. Hausch [16] considers only the case in which $\alpha = \frac{1}{2}$, while Campell and Levin [5] consider only the case in which $\alpha = \frac{1}{2}$ and the error is symmetric.

In the fully binary setting, the condition of informativeness of signals simply states that $q_1^P + p_2^P \geq 1$. Thus, informative signals are affiliated with v, and thus (by Lemma 19) also with each other. Thus, in the fully binary setting, informativeness of signals and full support are enough to apply our analysis.

7.1 Equilibrium Analysis

We illustrate the algorithm from Section 4 for a pure first-price auction ($\kappa = 1$) in the fully binary setting. We need to compute $F_1^P(b)$ and $F_2^P(b)$ for $P = Y, Z$.

By Lemma 3, at least one of $F_1^Y(b)$, $F_1^Z(b)$ will consist of an atom at $V(1,1)$. According to Lemma 3, the interval $\langle V(1,1), \bar{b}\rangle$ can be partitioned into smaller intervals, each the intersection of supports $W_y^Y \cap W_z^Z$ for two signals y, z. For binary signals, in addition to the interval consisting of only the point $V(1,1)$, there will be (at most) two such intervals, I_1 and I_2. The rightmost interval is the intersection of $W_2^Y \cap W_2^Z = \langle b_1, \bar{b}\rangle$. From Equation (6), after simple manipulations, we obtain for all $b \in I_2$:

$$F_2^Y(b) = 1 - \frac{\pi_Z(2)}{\pi(2,2)}\left(1 - \frac{V(2,2) - \bar{b}}{V(2,2) - b}\right), \qquad F_2^Z(b) = 1 - \frac{\pi_Y(2)}{\pi(2,2)}\left(1 - \frac{V(2,2) - \bar{b}}{V(2,2) - b}\right). \tag{17}$$

Without loss of generality, assume that $\pi_Y(2) \geq \pi_Z(2)$. Thus, $F_2^Y(b) \geq F_2^Z(b)$ for all $b \leq \bar{b} < V(2,2)$. At b_1, at least one of the two CDFs must become zero. It cannot be the CDF of Y, since then, at that point, the CDF of Z would be negative. (If $\pi_Y(2) = \pi_Z(2)$, then both CDFs will become zero simultaneously; therefore, they must become zero at $V(1,1)$. We can then compute $\bar{b}$, and the computation finishes.) The fact that $F_2^Z(b_1) = 0$ solves to

$$b_1 = V(2,2) - (V(2,2) - \bar{b})\frac{\pi_Y(2)}{\pi(2,1)}, \tag{18}$$

and $F_2^Y(b_1) = 1 - \frac{\pi_Z(2)}{\pi_Y(2)}$. Next, we look at the other interval $\langle V(1,1), b_1\rangle = W_2^Y \cap W_1^Z$. In this interval, the

CDFs will be of the form:

$$F_2^Y(b) = \left(\frac{\pi_Z(1)}{\pi(2,1)} - \frac{\pi_Z(2)}{\pi_Y(2)}\right) \cdot \frac{V(2,1)-b_1}{V(2,1)-b} - \frac{\pi(1,1)}{\pi(2,1)}, \qquad F_1^Z(b) = \frac{V(2,1)-b_1}{V(2,1)-b}. \tag{19}$$

By the fact that F_2^Y must become zero at $V(1,1)$, we get an equation that determines b_1:

$$\left(\frac{\pi_Z(1)}{\pi(2,1)} - \frac{\pi_Z(2)}{\pi_Y(2)}\right) \cdot \frac{V(2,1)-b_1}{V(2,1)-V(1,1)} = \frac{\pi(1,1)}{\pi(2,1)}. \tag{20}$$

Solving Equation 20 for b_1 gives us (after a few algebraic manipulations) that

$$b_1 = \alpha p_1^Z \cdot \frac{\pi_Y(2) - p_2^Y \pi_Z(2)}{\pi_Z(1)\pi_Y(2) - \pi_Z(2)\pi(2,1)}.$$

Having determined b_1, we can subsequently determine $\bar{b}$. This completes the equilibrium computation. In Figure 1, we show a sample plot of the distribution functions of the two bidders for some input parameters.

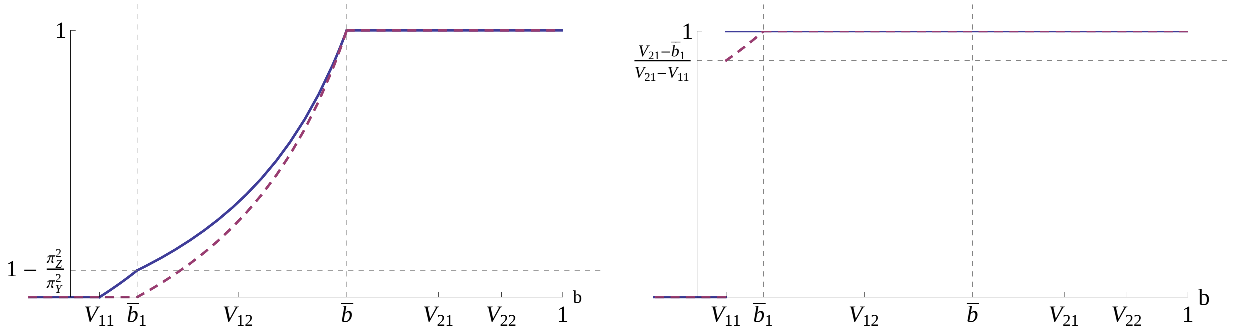

Figure 1: Sample cumulative distribution functions in the fully binary case with symmetric error. The parameter values are $\alpha = 0.7, p_Y = 0.75, p_Z = 0.6$. The left figure contains the distribution functions $F_2^P(b)$ of the two bidders conditional on getting the high signal, and the right figure those corresponding to the low signal.

In Appendix B we give a complete computation of the expected utilities of the two bidders in the above equilibrium. We also show how the computation gets simplified for special cases of the binary utility setting in which the signals can either identify only high-value items ("peaches") or only low-value items ("lemons").

8 Failure of the Linkage Principle in First-Price Auctions

The Linkage Principle states, informally, that in a common-value auction with symmetrically informed bidders, the auctioneer's revenue can never decrease by revealing all available information regarding the item's value. (See, e.g., [20].) The Linkage Principle does not hold in private-value settings, but it appears to be widely believed that the Linkage Principle would extend to common-value settings with asymmetrically informed bidders. In this section, we show that this belief is false: with ex ante asymmetrically informed bidders, the auctioneer's revenue can sometimes *strictly decrease* when revealing a signal to both bidders.

Theorem 14 (Failure of the Linkage-Principle) *The revenue of the auctioneer can decrease if he follows a policy of revealing a signal that is affiliated with the signals of the bidders, when bidders are asymmetrically informed.*

In fact, we show that this phenomenon can arise even in fully binary instances of a pure first-price auction, with symmetric errors. (See Section 7 for definitions.)

The auctioneer is considering revealing the value of another signal, which is also binary, and equal to v with probability $\hat{p}$. The signals of Y, Z and the auctioneer are independent conditional on v. Recall that we assume that p_Y, p_Z, $\hat{p}$ are commonly known to the bidders. Unless specified otherwise, we assume that Y is a priori at least as well informed as Z, i.e., $p_Y \geq p_Z$.

The equilibrium is explicitly characterized in Section 7.1, and the utility in Section B.1.3. However, notice that the calculations in Section 7.1 assume that bidder Y has the higher probability of seeing a high signal (rather than having a more accurate signal). It is straightforward that under our assumption $p_Y \geq p_Z$, bidder Y is more likely to see a high signal if and only if $\alpha \geq \frac{1}{2}$. Thus, in this case, we can apply the calculations of Section 7.1 directly, while in the case $\alpha < \frac{1}{2}$, the roles of Y and Z must be reversed. The resulting ex-ante expected utilities of the two bidders in the case $\alpha \geq \frac{1}{2}$ are

$$\begin{aligned} u^Y &= \frac{\alpha(1-\alpha)(1-p_Z)p_Z(2p_Y-1)}{\pi(1,1) + \left(1 - \frac{\pi_Z(2)}{\pi_Y(2)}\right)\pi(2,1)}, \\ u^Z &= \alpha(1-\alpha)\left(\frac{p_Z - p_Y}{\pi_Y(2)} + \frac{\pi_Z(2)}{\pi_Y(2)} \frac{(1-p_Z)p_Z(2p_Y-1)}{\pi(1,1) + \left(1 - \frac{\pi_Z(2)}{\pi_Y(2)}\right)\pi(2,1)}\right), \end{aligned} \tag{21}$$

while when $\alpha < \frac{1}{2}$:

$$\begin{aligned} u^Y &= \alpha(1-\alpha)\left(\frac{p_Y - p_Z}{\pi_Z(2)} + \frac{\pi_Y(2)}{\pi_Z(2)} \frac{(1-p_Y)p_Y(2p_Z-1)}{\pi(1,1) + \left(1 - \frac{\pi_Y(2)}{\pi_Z(2)}\right)\pi(1,2)}\right) \\ u^Z &= \frac{\alpha(1-\alpha)(1-p_Y)p_Y(2p_Z-1)}{\pi(1,1) + \left(1 - \frac{\pi_Y(2)}{\pi_Z(2)}\right)\pi(1,2)}. \end{aligned} \tag{22}$$

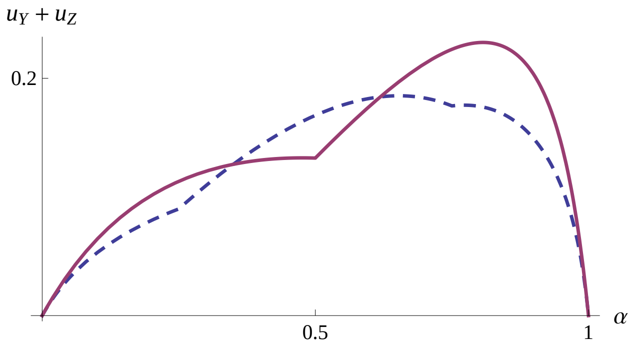

Figure 2: The solid line represents the total expected utility of two bidders as a function of α, when $p_Y = 0.95$ and $p_Z = 0.75$. The dashed line represents their total utility when the seller publicly announces another binary signal of accuracy $\hat{p} = 0.75$, i.e., each point of the dashed line is a convex combination of two points of the solid line. The Linkage Principle fails in the region around $\alpha = \frac{1}{2}$.

The point at which the Linkage Principle fails is the phase transition at $\alpha = \frac{1}{2}$. The equilibria in the two regions have very different behavior. Importantly, the total ex-ante expected utility of the bidders, $u^Y + u^Z$, becomes a non-concave function of α. For instance, Figure 2 depicts the total expected utility of the bidders as a function of α when $p_Y = 0.95$ and $p_Z = 0.75$.

When the auctioneer follows a revealing policy, the effects of revealing a signal can be captured by the bidders updating their prior on the item's value. Straightforward calculations show that when the auctioneer reveals a high signal to both bidders, they update their prior to $\hat{\alpha} = \frac{\alpha\hat{p}}{\alpha\hat{p}+(1-\alpha)(1-\hat{p})}$, while the signal accuracies remain the same. Similarly, when the seller reveals a low signal, the bidders' new prior is $\check{\alpha} = \frac{\alpha(1-\hat{p})}{\alpha(1-\hat{p})+(1-\alpha)\hat{p}}$.

Thus, a revelation policy creates a randomization over two auctions, one with prior $\hat{\alpha}$, the other with prior $\check{\alpha}$. The total utility of the two bidders is a convex combination of their utilities in those two auctions. Since the total utility is not concave, this convex combination can be strictly higher than the total utility at α (which is the convex combination of $\hat{\alpha}$ and $\check{\alpha}$). Figure 2 illustrates a case in which this happens.[3]

9 The Subtle Value of Extra Information

We now return to the question of the value of information to the bidders. Specifically, we consider a setting in which each bidder already has access to a signal, and a third signal is potentially available, meaning that it could be revealed to one or both (or neither) of the bidders.[4] The main result of the exploration in this

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[3] It is not difficult to show that the total utility *is* concave when $p_Y = p_Z$, i.e., the bidders are symmetric. This is not surprising, as the Linkage Principle is known to hold for symmetric bidders.

[4] We are interested in this question in part with an eye on how the owner of the third signal, who is different from the auctioneer, could best derive revenue from it.

section is that additional information exhibits several unexpected properties: it could have negative value or positive externalities.

At a first glance, when a bidder receives an extra signal, the result appears to be a multi-dimensional signal. Therefore, we first show that in several settings of interest, this multi-dimensional signal can be treated as a one-dimensional one under a carefully chosen complete order, such that the new signal satisfies both affiliation and informativeness.

The setting that we study in depth in this section is when the common value of the item is binary, and the signals are independent conditional on the true value. Let $S_1, \ldots, S_t$ be conditionally independent discrete signals, i.e., the signal S_j comes from the space $\Sigma_j = \{0, k_j\}$. Suppose that in the common-value auction setting, bidder Y had access to a subset J_Y of the above signals, while player Z had access to another disjoint subset J_Z. We show that such a setting falls into our setting of affiliated and informative signals. Thereto, we define a complete ordering of the multi-dimensional signals $y \in \times_{j \in J_Y} \Sigma_j$ and $z \in \times_{j \in J_Z} \Sigma_j$, by defining $y \geq y'$ if and only if $V_Y(y) \geq V_Y(y')$, i.e., by sorting signals by their conditional expected values.

First, we observe that these composite signals are also independent conditional on the true value, since each original signal was independent. Thus, once we have shown informativeness of the new signals, we can apply Lemma 20, which implies that the new one-dimensional signals are affiliated. Thus, we only need to show that the signals are informative, i.e., that $V(y, z)$ is increasing in both arguments, with respect to the ordering defined above. By Bayes' Law, for independent signals and true value $0, 1$,

$$V(y,z) = \frac{1}{1 + \frac{\mathbb{P}(v=0)}{\mathbb{P}(v=1)} \frac{\mathbb{P}(y|v=0)}{\mathbb{P}(y|v=1)} \frac{\mathbb{P}(z|v=0)}{\mathbb{P}(z|v=1)}}, \qquad V_Y(y) = \frac{1}{1 + \frac{\mathbb{P}(v=0)}{\mathbb{P}(v=1)} \frac{\mathbb{P}(y|v=0)}{\mathbb{P}(y|v=1)}},$$

and similarly for $V_Z(z)$. By definition of the orderings, $V_Y(y)$ is increasing in y, which, using the second formula, implies that $\frac{\mathbb{P}(y|v=1)}{\mathbb{P}(y|v=0)}$ is increasing in y, and similarly for bidder Z. Substituting into the first formula shows that $V(y, z)$ is increasing in each coordinate.

Therefore, when the common value is binary and all signals are independent conditional on the value v, our equilibrium characterization allows for a complete analysis of the value of extra signals to the bidders.

Even when the item's true value is not binary, the combination of multiple signals sometimes gives rise to a single-dimensional affiliated signal structure. One example is the information structure described by Einy et al. [10]. Suppose that the value is drawn from a finite subset W of $\mathbb{R}$ according to some prior probability distribution, and each signal S_j corresponds to an interval partition of the set of possible values: that is, if $\Sigma_j = \{1, \ldots, k_j\}$, then each value $\sigma_t^j \in \Sigma_j$ corresponds to a subset $W_t^j \subseteq W$, such that each W_t^j is an interval, the W_t^j are pairwise disjoint (for fixed j and distinct t, t'), consecutive, and a higher signal value corresponds to a higher interval. Then, the composition of any set of signals can be treated as a single-dimensional signal, where each value of the composite signal corresponds to the intersection of the intervals related to the values of each individual signal. The resulting intervals induce a natural order on the composite signals, and it is easy to show that the composite signals satisfy both the informativeness and affiliation property. Observe that as a player obtains more and more signals, the interval partition of his composite signal becomes finer and finer.

For the remainder of this section, to keep the analysis tractable, we restrict the analysis to a pure first-price auction in a fully binary setting with symmetric errors (see Section 7), which falls into the first type of setting: thus, a composition of signals can be treated as a single-dimensional signal. Even in this restricted setting, we exhibit surprising externality effects for extra information. Unless specified otherwise, we assume that Y is a priori at least as well informed as Z, i.e., $p_Y \geq p_Z$. We investigate the value of an additional conditionally independent binary signal with symmetric error $\hat{p}$. Such a signal could be allocated in four different ways:

1. Both bidders receive the signal,
2. Neither bidder receives the signal,
3. Bidder Y (the more informed bidder) receives the signal,
4. Bidder Z (the less informed bidder) receives the signal.

We show that for each of the outcomes (2)–(4), there are natural ranges of the parameters $\alpha, p_Y, p_Z, \hat{p}$ under which that outcome maximizes the combined welfare of the bidders. Even more strongly, for outcomes (3) and (4), there are parameter ranges under which *both bidders* strictly prefer that outcome over all other options.[5] This is quite remarkable: it means that bidders are willing to pay for their opponent to receive the signal. In the case of (3), this occurs when Y is already very well informed (and Z is not); by obtaining an accurate signal, Z would push both bidders' utilities close to 0. On the other hand, if Y receives the signal, Z obtains a small share of the added utility from Y's better information. In the case of (4), this occurs when α is very large. Y prefers having equally informed competition, and is willing to pay to ensure it. On the other hand, we are not aware of any setting in which the bidders' combined utility is maximized

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[5] This carries the implication that for such parameter ranges, a VCG-like selling mechanism cannot extract any revenue, even though it sells the signal to a bidder who has a strictly positive valuation for it. Hence, VCG-like mechanisms may not be well suited for selling signals for auctions.

by outcome (1). (Recall, though, that the example in Section 8 implied that there are cases in which the bidders' combined utility is higher under outcome (1) than under outcome (2).)

9.1 Perfect Information for Sale

As a first case, we consider the sale of a perfect signal, i.e., $\hat{p} = 1$. This case is simpler to analyze because a bidder purchasing a perfect signal has no more need for his original signal, so that we are in a setting in which both bidders receive binary signals, one of which is perfect. We identify the following behaviors for different regimes of the signals' qualities.

Proposition 15 (Strategic Ignorance) *Even if one bidder is already perfectly informed ($p_Y \to 1$), the less informed bidder still gets strictly positive utility as long as he is not completely uninformed and $\alpha > \frac{1}{2}$. Hence, the less informed bidder has a negative value for the extra signal. (If he were to receive it, both bidders would have zero utility.)*

Proof. First, if both bidders have perfect information, then they both get zero utility at equilibrium. Hence, when one bidder is already perfectly informed, the other bidder never gets positive gains from obtaining an additional perfect signal that is available for sale. We show the stronger statement that in fact, the value for obtaining the additional signal is strictly negative when the less informed bidder is not completely uninformed, and $\alpha > \frac{1}{2}$. This is equivalent to proving that bidder Z has positive utility when noone gets the additional perfect signal. If $\alpha > \frac{1}{2}$ then $\pi_Y(2) > \pi_Z(2)$. In that case, Equation (26) for the bidders' expected utilities simplifies to

$$u^Y = \alpha(1-\alpha) \cdot \frac{(1-p_Z)p_Z}{\pi_Z(1) - \pi_Z(2)(1-p_Z)}, \qquad u^Z = (1-\alpha) \cdot \frac{(2\alpha-1)(2p_Z-1)(1-p_Z)}{\pi_Z(1) - \pi_Z(2)(1-p_Z)}. \tag{23}$$

In particular, despite Y having perfect information, Z obtains positive utility so long as his private signal is informative but not perfect, i.e., so long as $p_Z \in (\frac{1}{2}, 1)$. ■

Proposition 15 implies that for $\alpha > \frac{1}{2}$, bidder Z would strictly prefer to remain strategically ignorant; a similar observation was made in the context of a slightly different common-value auction setting by Engelbrecht-Wiggans [11].

If $\alpha \leq \frac{1}{2}$, bidder Z's preference is not strict. For in that case, $\pi_Y(2) \leq \pi_Z(2)$, and the roles of Y and Z in the preceding calculations must be reversed. We obtain the following utilities:

$$u^Y = \alpha(1-\alpha) \cdot \frac{1-p_Z}{\pi_Z(2)}, \qquad\qquad u^Z = 0. \tag{24}$$

In this case, the less informed bidder always has zero utility, and therefore the value for extra information is simply zero. It is interesting to notice that Z's utility against a perfectly informed opponent undergoes a sharp transition at $\alpha = \frac{1}{2}$.

Proposition 16 (Agreeing on the More Informed Bidder) *There are regimes of the signals' qualities* p_Y, p_Z*, in which both* Y *and* Z *obtain strictly higher utility from* Y *getting the signal than from any other outcome.*

Before giving the proof of Proposition 16, we illustrate it with a concrete example (see Figure 3), with parameters $\alpha = 0.9, p_Y = 0.95$ and $p_Z = 0.75$. Intuitively, the outcome of the proposition is observed in regions where the item is a priori very likely to be valuable, and one of the two bidders is very well informed, while the other is only moderately informed. In Figure 3, one can observe the qualitative characteristics of the equilibria in all different outcomes.

If player Z gets the perfect signal (third plot of Figure 3), then essentially, both bidders are perfectly informed, and there is immense competition, with both bidders almost deterministically bidding the expected value conditional on the high signals. Thus, it is clear that neither bidder prefers this outcome. The more interesting effect appears in the comparison between the first two plots of Figure 3. Both bidders obtain positive utility only conditional on receiving their high signal, and their utilities can be calculated by examining only the point where the less informed bidder transitions from the support of his high signal to the support of his low signal. Observe that the actual point in the two cases is not significantly affected, while the probability that bidder Z wins against bidder Y by bidding that point increases by a larger amount. Hence, revealing the signal to Y yields a positive net effect on bidder Z's utility. Similarly, the utility of player Y from bidding that point does not decrease by much since the point only increases insignificantly. However, the probability that Y receives the high signal increases by a more significant amount; therefore, his overall utility also increases, leading to the conclusion that both bidders prefer bidder Y to receive the extra information.

Proof of Proposition 16. Suppose that the bidders initially have arbitrary signal qualities $p_Y > p_Z$, and

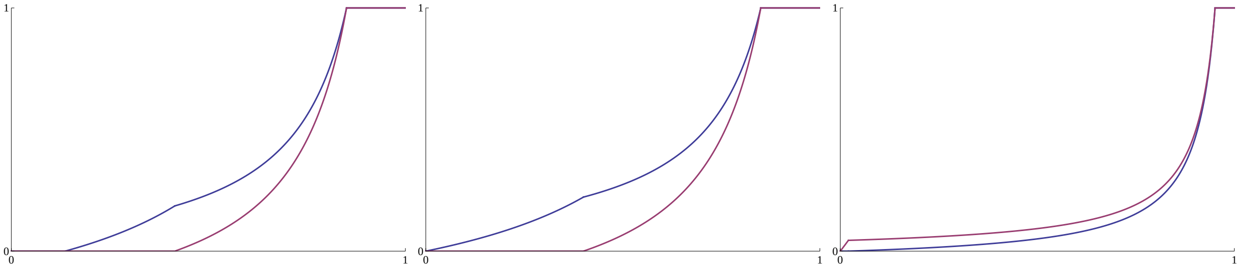

Figure 3: Equilibrium CDFs F_2^Y, F_2^Z for $\alpha = 0.9, p_Y = 0.95, p_Z = 0.75$. From left to right, we show the CDFs when neither bidder receives the perfect signal, bidder Y receives the perfect signal and bidder Z receives the perfect signal. The CDF of bidder Y is plotted with blue color.

that $\alpha > \frac{1}{2}$.[6]

If bidder Y acquires the signal, then bidder Z's utility is given by Equation (23). On the other hand, if bidder Z gets the signal, then he becomes the more informed bidder, and his utility is given by Equation (23), reversing the roles of Y and Z. Thus, bidder Z prefers bidder Y receiving the perfect information over himself receiving it iff the following holds:

$$(1-\alpha) \cdot \frac{(2\alpha - 1)(2p_Z - 1)(1 - p_Z)}{\pi_Z(1) - \pi_Z(2)(1 - p_Z)} > \alpha(1-\alpha) \cdot \frac{(1 - p_Y)p_Y}{\pi_Y(1) - \pi_Y(2)(1 - p_Y)}.$$

For Z to prefer the outcome of Y receiving the information, he must also prefer Y receiving it over noone receiving it. The utility of the latter outcome is given by the more general Equation (28).

While we cannot characterize exactly the region in which bidder Z prefers the outcome of Y receiving the signal, one particular setting of the signal qualities is when $\alpha = 0.9, p_Y = 0.95, p_Z = 0.75$. (This can be verified numerically). ■

Uniform Prior. The analysis becomes even simpler if we assume that the prior over the value is uniform: $\alpha = 1/2$. This setting was analyzed extensively by Hausch [16] since it satisfies his stronger condition for a symmetric equilibrium. In this setting, the equilibrium utility of the bidders simplifies to $u^Y = \frac{V(2,2) - p_Z}{2}$ and $u^Z = \frac{V(2,2) - p_Y}{2}$. In this setting, the value of extra information behaves more as expected: each bidder has positive value of $\frac{1 - V(2,2)}{2}$ for obtaining a perfect signal, and information has a negative externality on the opponent, since his utility becomes zero. The total utility of the two bidders is maximized either by allocating the signal exclusively to the more informed bidder, or by not allocating it at all. The second option maximizes the combined bidders' utility when $p_Y > \frac{1 - p_Z}{2p_Z - 1}$.

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[6] When $\alpha \leq \frac{1}{2}$, the sale of a perfect signal to Y will always result in zero utility for Z, so Z can never prefer that outcome.

9.2 One a priori Uninformed Bidder

Next, we investigate the case when one or both bidders are a priori completely uninformed, which is equivalent to having a signal with $p_P = \frac{1}{2}$ (which is a uniformly random bit, independent of the item's value).

An interesting phenomenon of positive externalities arises when one bidder, say, Z, is initially uninformed, $p_Z = \frac{1}{2}$, and the additional signal is of the same quality as Y's signal, i.e., $\hat{p} = p_Y =: p$. We prove the following:

Proposition 17 (Agreeing on the Less Informed Bidder) *If Z is uninformed, α is large enough, and $p = \hat{p} = p_Y < \alpha$ is large enough, then both bidders strictly prefer for Z to obtain the signal exclusively.*

In the same setting, for any $\alpha > \frac{1}{2}$, so long as $p < \alpha$, bidder Y prefers Z obtaining the signal exclusively compared to shared access or neither bidder obtaining it.

Intuitively, the proposition (whose proof we give below) shows that if a bidder is only moderately well informed compared to the prior, then he prefers competing with an equally informed opponent, as opposed to a completely uninformed one.

The intuition is that free-riding will always occur at equilibrium, in the sense that the uninformed bidder is bound to win with positive probability, and the support of the strategy of the informed bidder conditional on the high signal will overlap with the strategy of the uninformed one. The informed bidder is not willing to lose with high probability against the uninformed one and therefore is not shading his bid by much.

On the other hand, when the opponent receives an additional independent signal, then his bid contains significant and disjoint new information. At equilibrium, the informed bidder will lose only conditional on the new signal being high. Hence, whenever he is losing, he knows that the other bidder got a high signal, which pushes the expected value even higher. This allows the initially informed bidder to shade his bid even more. Technically, the upper bounds of the two bidder strategies will be lower (see Figure 4) when the uninformed bidder becomes informed, thereby leading to higher expected utility.

Proof of Proposition 17. If neither bidder obtains the signal, then according to Equation (26) the bidders' utilities simplify to $\alpha(1-\alpha)(2p-1)$, and 0, respectively. If Y obtains the signal, then applying Theorem 4 of [12] (or using our general recurrence) shows that the utility of Y is now $\alpha(1-\alpha) \cdot (2p-1)(1+2p(1-p))$, while the utility of Z remains 0.

When Z receives the signal, the item auction becomes symmetric and has a unique symmetric equilibrium. Using our analysis with $b \to V(1,1)$ (or the analysis for symmetrically informed bidders by Wang [39]) then gives that both bidders' utilities are $\alpha(1-\alpha)\frac{(1-p)p(2p-1)}{\alpha(1-p)^2+(1-\alpha)p^2}$.

Finally, when both Y and Z receive the signal, we can use the following reasoning. If the common signal is revealed to be high, then the conditional expected value of the item changes to $\frac{\alpha p}{\alpha p+(1-\alpha)(1-p)}$; subject to this modified prior, bidder Z is now again uninformed, while Y receives another signal which is correct with probability p. Hence, the utilities can be calculated as in the previous case of one uninformed bidder. Similar reasoning applies when the revealed signal is low. By combining both cases, we find that Z's expected utility is 0, while Y has expected utility $\alpha(1-\alpha)(2p-1) \cdot \frac{p(1-p)}{\pi_Y(1)(1-\pi_Y(1))}$. The utilities are summarized in the following table.

	u^Y	u^Z
Both bidders win	$\alpha(1-\alpha)(2p-1) \cdot \frac{p(1-p)}{\pi_Y(1)(1-\pi_Y(1))}$	0
No bidder wins	$\alpha(1-\alpha)(2p-1)$	0
Bidder Y wins	$\alpha(1-\alpha)(2p-1) \cdot (1+2p(1-p))$	0
Bidder Z wins	$\alpha(1-\alpha)(2p-1) \cdot \frac{p(1-p)}{\alpha(1-p)^2+(1-\alpha)p^2}$	$\alpha(1-\alpha)(2p-1) \cdot \frac{p(1-p)}{\alpha(1-p)^2+(1-\alpha)p^2}$

It is clear that Z prefers the outcome of winning the signal exclusively. However, Y's preferences are quite remarkable. First, Y always prefers receiving the signal over neither bidder receiving it. Second, Y always prefers neither bidder receiving the signal over both bidders receiving it. This can be seen because both $\pi_Y(1)$ and $1-\pi_Y(1)$ are convex combinations of p and $1-p$; hence $\frac{p(1-p)}{\pi_Y(1)(1-\pi_Y(1))} < 1$.

The interesting cases arise regarding Y's utility resulting from Z receiving the signal. Whenever $\alpha > \frac{1}{2}$ and $p \in (\frac{1}{2}, \alpha)$, Y strictly prefers Z having the signal over neither bidder having it. In other words, when he himself is only moderately well informed compared to the prior, a bidder prefers competing with an equally informed opponent, as opposed to a completely uninformed one. In Figure 4, we show the equilibrium CDFs for such a setting, before and after Z obtains the new signal. The figure shows that when the less informed bidder obtains a new signal, both bidders shade their bids even more at equilibrium and thereby get higher expected utility.

Next, consider the ratio between Y's utility when he receives the signal and when Z receives the signal: $\frac{(1+2p(1-p))\cdot(\alpha(1-p)^2+(1-\alpha)p^2)}{p(1-p)}$. As $\alpha \to 1$, this ratio goes to $\frac{(1+2p(1-p))\cdot(1-p)}{p}$. For large enough p, the ratio becomes less than 1. Thus, if the item is sufficiently likely to be valuable, and the signals are less informative than the prior, but still sufficiently informative, Y strictly prefers for Z to receive the signal over himself receiving it. In fact, the parameter setting of Figure 4 shows such a case. ■

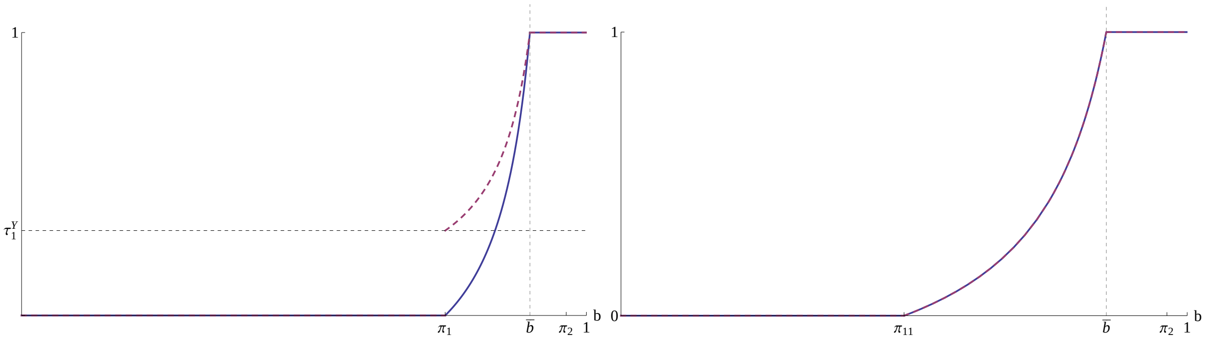

Figure 4: The equilibrium CDFs F_2^Y, F_2^Z for two bidders. In both figures, $\alpha = .9$ and $p_Y = .75$. In the left figure, Z is uninformed; conditional on receiving the low signal, bidder Y then deterministically bids π_1. In the right figure, Z receives an independent signal of quality $p_Z = p_Y$, and both bidders now shade their bids more.

10 When is a high signal good news?

In this section we examine the question when receiving a higher signal implies a higher expected utility for a bidder. We establish that the following two conditions are sufficient:

1. The value v of the item is binary, and $v = 1$ with probability α.
2. The bidders' signals are independent conditional on v.

Under these assumptions, the expected value conditioned on a signal can be simplified to $V_Y(y) = \mathbb{P}(v = 1 | S_Y = y)$. Also, under these assumptions, Lemma 20 in Appendix A shows that affiliation of the signals is directly implied by informativeness. In other words, if $V_Y(y)$ and $V_Z(z)$ are both increasing in y (resp., z), then the signals of the two bidders are affiliated.

When the two conditions hold, the distributions of the bidders' signals can be very succinctly described, as follows: Conditional on the $v = 0$, bidder P receives the signal $\sigma \in \Sigma_P$ with probability q_σ^P. When $v = 1$, he receives the signal $\sigma \in \Sigma_P$ with probability p_σ^P.

We begin by giving alternative expressions for u_σ^P, taking advantage of the more restrictive signal structure:

$$\begin{aligned} u_y^Y(b) = V_Y(y) \cdot \sum_{z \in \Sigma_Z} p_z^Z \cdot &\left((1 - \kappa b) \cdot F_z^Z(b) - (1 - \kappa) \int_{\underline{b}_z^Z}^{b} \mu dF_z^Z(\mu) \right) \\ + (1 - V_Y(y)) \cdot \sum_{z \in \Sigma_Z} q_z^Z \cdot &\left(-\kappa b F_z^Z(b) - (1 - \kappa) \int_{\underline{b}_z^Z}^{b} \mu dF_z^Z(\mu) \right) \\ = V_Y(y) \cdot h_Y^+(b) + (1 - V_Y(y)) h_Y^-(b), & \end{aligned} \tag{25}$$

where we write

$$
\begin{aligned}
h_Y^+(b) &= \sum_{z\in\Sigma_Z} p_z^Z \cdot \left((1-\kappa b)\cdot F_z^Z(b) - (1-\kappa)\int_{\underline{b}_z^Z}^{b} \mu dF_z^Z(\mu)\right)\\
&= \sum_{z\in\Sigma_Z} p_z^Z \cdot \left((1-b)\cdot F_z^Z(b) + (1-\kappa)\int_{\underline{b}_z^Z}^{b} F_z^Z(\mu)d\mu\right) \;\geq\; 0, \qquad \text{and}\\
h_Y^-(b) &= -\sum_{z\in\Sigma_Z} q_z^Z \cdot \left(\kappa b F_z^Z(b) + (1-\kappa)\int_{\underline{b}_z^Z}^{b} \mu dF_z^Z(\mu)\right) \leq 0.
\end{aligned}
$$

Observe that neither $h_Y^+(b)$ nor $h_Y^-(b)$ depend on the signal y of bidder Y. To see that $h_Y^+(b) \geq 0$, notice that the second term $(1-\kappa)\int_{\underline{b}_z^Z}^{b} F_z^Z(\mu)d\mu$ is always non-negative because $F_z^Z(\mu) = 0$ whenever $b < \underline{b}_z^Z$; the first term is non-negative because $b \leq \bar{b} \leq V(k_Y, k_Z) \leq 1$.

Using this formulation of expected utility, we can show that a higher signal always guarantees at least the same expected utility as a lower signal.

Lemma 18 (Sorted Utilities) *Let P be a bidder, and $\sigma, \sigma-1 \in \Sigma_P$ be signals. Then, $u_\sigma^P \geq u_{\sigma-1}^P$. So a higher signal is always good news for a bidder.*

Proof. Without loss of generality, let $P = Y$. Let b be an arbitrary bid in the support of F_{y-1}^Y. We will show that Y, when receiving the signal y, can bid b and obtain utility at least u_{y-1}^Y; this implies that with his best bid, he must obtain at least the same utility.

By the informativeness of the signals, $V(y,z) > V(y-1,z)$ for any z. Consider now Equation (25). All the terms except $V_Y(y)$ and $1 - V_Y(y)$ are independent of bidder Y's signal y, and constant for fixed b. The first term is non-negative, while the second is non-positive. Thus, the convex combination with weight $V_Y(y)$ on the first term is at least as large as the one with $V_Y(y-1)$, and bidding b with signal j must give Y utility at least $u_{y-1}^Y(b)$. ■

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A Technical Lemmas and Proofs

We restate lemmas and theorems for convenience before proving them.

A.1 Properties of Affiliation

Lemma 19 is an analogue of an observation of de Castro [9, p. 14], phrased for discrete distributions.

Lemma 19 *When the signals are independent conditional on the true value v, affiliation of the signals with the value v implies affiliation of the signals with each other.*

Proof. For notational convenience in the following derivation, we denote with 0 the event that $v = 0$ and with 1 the event that $v = 1$. Affiliation of the signals with the value means that for any $v \le v'$ and any $\sigma \le \sigma' \in \Sigma_P$:

$$\mathbb{P}\left(v|\sigma\right) \cdot \mathbb{P}\left(v'|\sigma'\right) \ge \mathbb{P}\left(v|\sigma'\right) \cdot \mathbb{P}\left(v'|\sigma\right).$$

We will show that the latter implies that for any $y \le y' \in \Sigma_Y$ and $z \le z' \in \Sigma_Z$:

$$\pi(z|y) \cdot \pi(z'|y') \ge \pi(z'|y) \cdot \pi(z|y').$$

We begin by writing the individual products in terms of conditional probabilities for values v.

$$\begin{aligned}
\pi(z|y) \cdot \pi(z'|y') &= \left(\sum_{v} \mathbb{P}\left(z|v\right) \cdot \mathbb{P}\left(v|y\right)\right) \cdot \left(\sum_{v'} \mathbb{P}\left(z'|v'\right) \cdot \mathbb{P}\left(v'|y'\right)\right) \\
&= \sum_{v,v'} \mathbb{P}\left(z|v\right) \cdot \mathbb{P}\left(v|y\right) \cdot \mathbb{P}\left(z'|v'\right) \cdot \mathbb{P}\left(v'|y'\right) \\
&= \frac{1}{2} \cdot \left(\sum_{v,v'} \mathbb{P}\left(z|v\right) \cdot \mathbb{P}\left(v|y\right) \cdot \mathbb{P}\left(z'|v'\right) \cdot \mathbb{P}\left(v'|y'\right) + \sum_{v,v'} \mathbb{P}\left(z|v'\right) \cdot \mathbb{P}\left(v'|y\right) \cdot \mathbb{P}\left(z'|v\right) \cdot \mathbb{P}\left(v|y'\right)\right) \\
&= \frac{1}{2} \cdot \sum_{v,v'} \left(\mathbb{P}\left(z|v\right) \cdot \mathbb{P}\left(v|y\right) \cdot \mathbb{P}\left(z'|v'\right) \cdot \mathbb{P}\left(v'|y'\right) + \mathbb{P}\left(z|v'\right) \cdot \mathbb{P}\left(v'|y\right) \cdot \mathbb{P}\left(z'|v\right) \cdot \mathbb{P}\left(v|y'\right)\right).
\end{aligned}$$

Now, using that $\mathbb{P}\left(z|v\right) = \pi_Z(z) \cdot \frac{\mathbb{P}(v|z)}{\mathbb{P}(v)}$ (and similarly for z', v'), we can rewrite the product as

$$\frac{\pi_Z(z) \cdot \pi_Z(z')}{2} \cdot \sum_{v,v'} \frac{\mathbb{P}\left(v|z\right) \cdot \mathbb{P}\left(v|y\right) \cdot \mathbb{P}\left(v'|z'\right) \cdot \mathbb{P}\left(v'|y'\right) \ + \ \mathbb{P}\left(v'|z\right) \cdot \mathbb{P}\left(v'|y\right) \cdot \mathbb{P}\left(v|z'\right) \cdot \mathbb{P}\left(v|y'\right)}{\mathbb{P}\left(v\right) \cdot \mathbb{P}\left(v'\right)}.$$

Completely analogous derivations show that

$$\pi(z'|y) \cdot \pi(z|y') = \frac{\pi_Z(z) \cdot \pi_Z(z')}{2} \cdot \sum_{v,v'} \frac{\mathbb{P}\left(v|z'\right) \cdot \mathbb{P}\left(v|y\right) \cdot \mathbb{P}\left(v'|z\right) \cdot \mathbb{P}\left(v'|y'\right) \ + \ \mathbb{P}\left(v'|z'\right) \cdot \mathbb{P}\left(v'|y\right) \cdot \mathbb{P}\left(v|z\right) \cdot \mathbb{P}\left(v|y'\right)}{\mathbb{P}\left(v\right) \cdot \mathbb{P}\left(v'\right)}.$$

Writing the difference and regrouping now gives us that

$$\begin{aligned}
&\pi(z|y) \cdot \pi(z'|y') - \pi(z'|y) \cdot \pi(z|y') \\
=&\frac{\pi_Z(z) \cdot \pi_Z(z')}{2} \cdot \sum_{v,v'} \frac{(\mathbb{P}(v|z) \cdot \mathbb{P}(v'|z') - \mathbb{P}(v|z') \cdot \mathbb{P}(v'|z)) \cdot (\mathbb{P}(v|y) \cdot \mathbb{P}(v'|y') - \mathbb{P}(v'|y) \cdot \mathbb{P}(v|y'))}{\mathbb{P}(v) \cdot \mathbb{P}(v')} \\
\geq& 0,
\end{aligned}$$

where the last inequality follows from affiliation of signals with values, which implies that the two factors in the numerator are either both non-negative or both non-positive. ■

In Section 10, we also need the fact that informativeness (which captures monotonicity of expected value) is equivalent with affiliation with the value.

Lemma 20 *If the value of the item is binary, then informativeness of a signal is equivalent to affiliation between the signal and the value.*

Proof. For notational convenience in the following derivation, we assume that the item's value is either 0 or 1, and we denote with 0 the event that $v = 0$ and with 1 the event that $v = 1$. We need to show that for any $\sigma' \leq \sigma \in \Sigma$, we have $\mathbb{P}(1|\sigma)\mathbb{P}(0|\sigma') \geq \mathbb{P}(0|\sigma)\mathbb{P}(1|\sigma')$, which, using that $\mathbb{P}(0|\sigma) = 1 - \mathbb{P}(1|\sigma)$, is equivalent to $\mathbb{P}(1|\sigma) \geq \mathbb{P}(1|\sigma')$. Observe that the latter inequality follows from informativeness, since the quantities on each side correspond to the expected value conditional on the corresponding signals. ■

A.2 Proof of Lemma 3

Lemma 3 *If Assumptions 1, 2 and 3 hold, then any (mixed) Nash Equilibrium F^P_σ of the common value hybrid auction satisfies the following:*

1. $\underline{b}^Y = \underline{b}^Z = V(1,1)$. *Thus, the union of supports is the same for both bidders, and the lowest bid in the support of either bidder is the value of the item conditioned on the lowest signal for both bidders.*
2. *At least one of the two bidders deterministically bids $V(1,1)$ when receiving the lowest signal 1. Both bidders have expected utility 0 when receiving the lowest signal, i.e., $u^Y_1 = u^Z_1 = 0$.*
3. *The support of the distribution F^P_σ for $\sigma \in \Sigma_P$ is of the form $W^P_\sigma = \langle b^P_{\sigma-1}, b^P_\sigma \rangle$. In particular, for a fixed bidder, the supports of his distributions with distinct signals are consecutive intervals, in the region of winning bids.*

Proof of Lemma 3. Without loss of generality, assume that $\underline{b}^Y \leq \underline{b}^Z$; if $\underline{b}^Y = \underline{b}^Z$ and both bidders bid this value with positive probability, assume further that Z wins the tie breaking with positive probability. By Lemma 2, the supports are monotone in the region $(\underline{b}^Z, \bar{b}]$.

Let $\Sigma_Y^0 = \{y \mid W_y^Y \cap [0, \underline{b}^Z] \neq \emptyset\}$ be the set of signals of bidder Y under which at least one bid in $[0, \underline{b}^Z]$ is possible. Let $\hat{y} = \max \Sigma_Y^0$ be the maximum such signal. First, we will show that

$$V(\hat{y}, 1) \ \leq \ \underline{b}^Z \ \leq \ \sum_{y \in \Sigma_Y^0} \alpha_y \cdot V(y, 1),$$

for some strictly positive multipliers α_y with $\sum_{y \in \Sigma_Y^0} \alpha_y = 1$. The maximality of $\hat{y}$ and the strict monotonicity of the expected value (Assumption 1) will then imply that $\Sigma_Y^0 = \{\hat{y}\}$ and that $\underline{b}^Z = V(\hat{y}, 1)$. We will then show that $\hat{y} = 1$.

- First, suppose that $\underline{b}^Z < V(\hat{y}, 1)$. Let b' be a bid slightly (infinitesimally) above $\underline{b}^Z$. If Y bids b', his expected utility is strictly positive, because he wins with positive probability and obtains positive utility (at least $V(\hat{y}, 1) - b'$) when winning. Since any bid $b < \underline{b}^Z$ deterministically loses for Y (resulting in utility 0), Y cannot make any such bid at equilibrium, so $\underline{b}^Y = \underline{b}^Z$. Similarly, if Z does not have positive probability of bidding $\underline{b}^Z$, then a bid of $\underline{b}^Y$ wins with probability 0, again giving Y utility 0. Finally, if Z has positive probability of bidding $\underline{b}^Z$, then Z wins with positive probability when both bid $\underline{b}^Z$, by definition of Y, Z. By bidding b' instead, Y thus increases his winning probability by a positive amount, and obtains positive utility conditioned on the new winning scenarios. Thus, in all cases, Y can increase his expected utility, and we obtain a contradiction to our assumption.

- Next, consider any signal $y > \hat{y}$. Assume that Z receives the signal 1 and makes a bid $b > \underline{b}^Z$. By definition of $\hat{y}$, we have that $\underline{b}^Z \notin W_y^Y$. Therefore, $F_y^Y(b) \to 0$ as $b \to \underline{b}^Z$. In particular,

$$\begin{aligned} u_1^Z(b) &\to \sum_{y \in \Sigma_Y^0} \alpha_y \cdot V(y, 1) \qquad \text{as } b \to \underline{b}^Z; \text{ here} \\ \alpha_y &= \lim_{b \to \underline{b}^Z} \mathbb{P}\,(Y \text{ sees signal } y | Y \text{ bids less than } b \text{ and } Z \text{ sees signal } 1) \\ &= \lim_{b \overset{+}{\to} \underline{b}^Z} \frac{\pi(y|1) \cdot F_y^Y(b)}{\sum_{y'} \pi(y'|1) \cdot F_{y'}^Y(b)} \ > \ 0 \end{aligned}$$

  by the full support assumption (Assumption 3) and monotonicity. By taking the sum over all $y \leq \hat{y}$, and exchanging with the limit, we also get that $\sum_{\sigma_y \leq \hat{y}} \alpha_y = 1$.

Thus, if $\underline{b}^Z > \sum_{y \in \Sigma_Y^0} \alpha_y \cdot V(y,1)$, by bidding at (or infinitesimally above) $\underline{b}^Z$, bidder Z would obtain negative utility, a contradiction. This completes the proof of the second bound.

Because $V(1,1) < V(2,1) < \cdots < V(\hat{y},1)$ by Assumption 1, we get that $\sum_{y \in \Sigma_Y^0} \alpha_y \cdot V(y,1) \leq V(\hat{y},1)$, so in fact, $\sum_{y \in \Sigma_Y^0} \alpha_y \cdot V(y,1) = V(\hat{y},1)$. This is possible only when $\alpha_{\hat{y}} = 1$, which in turn implies that $\Sigma_Y^0 = \{\hat{y}\}$. Thus, $\underline{b}^Z = V(\hat{y},1)$.

By definition, when bidder Z bids $\underline{b}^Z$, he can only win against bidder Y with a signal of $\hat{y}$. In that case, the conditional utility is 0, so Z's expected utility from bidding $\underline{b}^Z$ is 0. If $\underline{b}^Z \notin W_1^Z$, we can obtain the same result by considering $b \to \underline{b}^Z$ from above; in that case, the probability of winning against any other bid goes to 0, so the expected utility converges to 0. In either case, we have derived that $u_1^Z = 0$.

In turn, this implies that $\underline{b}^Y = V(\hat{y},1)$ (and thus also that $u_{\hat{y}}^Y = 0$). For if $\underline{b}^Y < V(\hat{y},1)$, then Z could obtain positive utility (when receiving signal 1) by bidding strictly between $\underline{b}^Y$ and $V(\hat{y},1)$. In summary, we have shown that

$$\underline{b}^Y = \underline{b}^Z = V(\hat{y},1), \text{ and } u_1^Z = u_{\hat{y}}^Y = 0.$$

Next, we show that $\hat{y} = 1$. Unfortunately, this does not follow directly from Lemma 2, as Lemma 2 implies monotonicity of signals of Y only over the region of winning bids for Z. First, if F_2^Z had an atom at $V(\hat{y},1)$ (i.e., Z, when receiving the signal 2, had strictly positive probability of bidding $V(\hat{y},1)$), then Y with signal $\hat{y}$ could guarantee strictly positive utility by bidding slightly above $V(\hat{y},1)$, instead of his utility of 0. Thus, $\lim_{b \to V(\hat{y},1)} F_2^Z(b) = 0$. Because the supports of the distributions F_1^Y and F_2^Y cannot overlap, this implies either that there exists a $\hat{b} > V(\hat{y},1)$ with $F_2^Y(\hat{b}) = 0$, or that bidder Z deterministically bids $V(\hat{y},1)$ when receiving the signal 1.

Now consider bidder Y receiving signal 1. By monotonicity (Lemma 2), applied to bids $b > V(\hat{y},1)$, the lower bound of W_1^Y must equal $V(\hat{y},1)$. We now distinguish two cases:

- If there exists a $\hat{b} > V(\hat{y},1)$ with $F_2^Y(\hat{b}) = 0$, then any bid below $\hat{b}$ will win against Z having signal 1 with positive probability, and against Z having signal 2 with probability 0. Thus, bidder Y can only win against signal 1 with bids $b \in (V(\hat{y},1), \hat{b})$.
- If bidder Z deterministically bids $V(\hat{y},1)$ when receiving the signal 1, then as $b \to V(\hat{y},1)$, the probability that Y wins against any signal other than 1 for Z goes to 0, while the probability of winning against Z having signal 1 stays at 1.

In either case, for sufficiently small b, the expected value of the item, conditioned on Y winning with a bid b, goes to $V(1,1)$. On the other hand, the payment is at least $V(\hat{y},1)$. Thus, unless $\hat{y}=1$, the informativeness of the signals would imply strictly negative expected utility for Y. We conclude that $\hat{y}=1$.

Thus we have shown that $\underline{b}^Y = \underline{b}^Z = V(1,1)$ and $\Sigma_Y^0 = \{1\}$. Finally, we show that at least one bidder must bid $V(1,1)$ deterministically when receiving signal 1. If not, then $W_1^Y \cap W_1^Z$ would contain some open interval (b_1, b_2). Any bid b in this interval would win with positive probability against a signal 1 by the other bidder, and with probability 0 against other signals (by Lemma 2). As $b > b_1 \geq V(1,1)$, such a bid would give negative expected utility. ■

B Expected Utility with Binary Signals in the First-Price Auction

In this section, we complete the analysis for auctions with binary signals. In Section 7, we calculated the equilibrium bidding functions; we now give explicit formulas for the expected utility of the two bidders.

First observe that the expected value of the item conditional on the high signal is $\frac{\alpha p_2^P}{\alpha p_2^P+(1-\alpha)q_2^P}$; conditional on the low signal, the value is: $\frac{\alpha p_1^P}{\alpha p_1^P+(1-\alpha)q_1^P}$.

To calculate the two bidders' utilities, we first substitute the value of b_1 into Equation (3) for bidder Y's utility to obtain his utility conditioned on the high signal. Then, to calculate bidder Z's utility, we notice that when bidding $\bar{b}$, each bidder wins deterministically, so $u_2^Y = u_2^Y(\bar{b}) = V_Y(2) - \bar{b}$, and $u_2^Z = V_Z(2) - \bar{b}$. Thus, $u_2^Z = u_2^Y + (V_Z(2) - V_Y(2))$. With some algebraic manipulations, we then obtain that

$$\begin{aligned}
u^Y &= \pi_Y(2)u_2^Y(b_1)\\
&= \alpha p_1^Z \pi_Y(2) \cdot \frac{p_2^Y \pi_Z(1) - \pi(2,1)}{\pi_Z(1)\pi_Y(2) - \pi_Z(2)\pi(2,1)}\\
&= \alpha(1-\alpha)p_1^Z q_1^Z \pi_Y(2) \cdot \frac{p_2^Y - q_2^Y}{\pi_Z(1)\pi_Y(2) - \pi_Z(2)\pi(2,1)},\\
u^Z &= \pi_Z(2)u_2^Z(b_1)\\
&= \pi_Z(2)(u_2^Y(b_2) + (V_Z(2) - V_Y(2)))\\
&= \alpha \cdot \frac{(\pi_Y(2) - p_2^Y \pi_Z(2)) \cdot (p_2^Z \pi_Y(2)\pi_Z(1) - \pi_Z(2)\pi(2,1))}{\pi_Y(2)(\pi_Z(1)\pi_Y(2) - \pi_Z(2)\pi(2,1))}\\
&= \alpha(1-\alpha) \cdot (\pi(2,1) - (1-\alpha)q_2^Z(p_2^Y - q_2^Y)) \cdot (\pi_Y(2) - q_2^Y \pi_Z(2)) \cdot \frac{p_2^Z - q_2^Z}{\pi_Y(2)(\pi_Z(1)\pi_Y(2) - \pi_Z(2)\pi(2,1))}.
\end{aligned} \tag{26}$$

B.1 Special Cases

We next investigate several special cases, which allow further simplification of the formulas. These special cases correspond to cases studied in the literature in the past (lemons, peaches) and cases analyzed in more depth later (symmetric error).

B.1.1 Identifying only Lemons

A special case studied in past work (e.g., [1], who consider it for second-price auctions) is when the binary signal is only helpful in identifying *lemons*, i.e., items of value 0. In other words, if the item is valuable, then each signal (high or low) is equally likely, whereas when the item is worthless, the low signal is more likely. Formally, this means that $p_2^P = p_1^P = \frac{1}{2}$, and $q_1^P > \frac{1}{2}$ for both bidders P.

In this case, in our computations with $\pi_Y(2) > \pi_Z(2)$, bidder Y is the less informed bidder. Intuitively, the reason is that seeing a high signal is a sign of a less accurate "lemon detector." More formally, since $\pi_P(2) = \frac{\alpha}{2} + (1-\alpha)(1-q_1^P)$, the assumption that $\pi_Y(2) > \pi_Z(2)$ is equivalent to $q_1^Y < q_1^Z$. Assigning Y the role of less informed bidder and simplifying gives us the following expected utilities:

$$\begin{aligned} u^Y &= \alpha(1-\alpha)\frac{1}{2}q_1^Z\left(\frac{\alpha}{2} + (1-\alpha)(1-q_1^Y)\right) \cdot \frac{q_1^Y - \frac{1}{2}}{\pi_Z(1)\pi_Y(2) - \pi_Z(2)\pi(2,1)}, \\ u^Z &= \alpha(1-\alpha) \cdot (\pi(2,1) - (1-\alpha)(1-q_1^Z)(q_1^Y - \frac{1}{2})) \cdot \frac{((1-\alpha)(q_1^Z - q_1^Y) + q_1^Y \pi_Z(2))(q_1^Z - \frac{1}{2})}{\pi_Y(2)(\pi_Z(1)\pi_Y(2) - \pi_Z(2)\pi(2,1))}. \end{aligned} \tag{27}$$

Whenever both bidders' lemon identifiers are perfect, i.e., $q_1^Y = q_1^Z = 1$, the setting is symmetric, and the utility of both bidders is $\frac{\alpha(1-\alpha)}{4-3\alpha}$. This expression, and thus the bidders' utilities, is maximized when $\alpha = 2/3$.

B.1.2 Identifying only Peaches

An apparently similar situation arises when the signal can only identify *peaches*, i.e., valuable items. In that case, $q_1^P = q_2^P = \frac{1}{2}$. One can derive an expression somewhat similar to Equation (27) for this case. If the peach identifiers are perfect, i.e., $p_2^Y = p_2^Z = 1$, we obtain a symmetric setting in which both bidders have expected utility 0.

B.1.3 Symmetric Error

For symmetric error (see Definition 13), the expressions for utilities simplify to

$$
\begin{aligned}
u^{Y} &= \alpha(1-\alpha)\pi_Y(2)\frac{(1-p_Z)p_Z(2p_Y-1)}{\pi_Z(1)\pi_Y(2)-\pi_Z(2)\pi(2,1)},\\
u^{Z} &= \alpha(1-\alpha)\left(\frac{p_Z-p_Y}{\pi_Y(2)}+\frac{\pi_Z(2)(1-p_Z)p_Z(2p_Y-1)}{\pi_Z(1)\pi_Y(2)-\pi_Z(2)\pi(2,1)}\right).
\end{aligned}
\tag{28}
$$

We will make use of this expression in Sections 8 and 9.