Semi-Parametric Efficient Policy Learning with Continuous Actions

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Optimal Pricing from Historical/Observational Data

We are given historical data of demand and price from a company.

**Goal:** Find the optimal price point based on the data.

**Approach:** Estimate demand function $Q(p)$ then optimize revenue $p \cdot Q(p)$.

**Conclusion:** Increasing price increases demand!

**Problem:** Demand increases in winter and price anticipates demand.
Optimal Pricing from Observational Data

We are given historical data of demand and price from a company

**Goal:** Find a new optimal price point based on the data

**Approach:** Estimate demand function $Q(p, x)$ then optimize expected revenue $V(p) = E_X[p \cdot Q(p, X)]$

**Idea:** Introduce confounder (the season) into regression

\[
Y = \alpha_0 \cdot P + \beta_0(X) + \epsilon
\]

- $Y$: demand
- $P$: price
- $X$: confounders
- $\epsilon$: noise
Optimal Pricing from Observational Data

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**Goal:** Find a new optimal price point based on the data

**Approach:** Estimate demand function $Q(p, x)$ then optimize expected revenue

$$V(p) = E_X[p \cdot Q(p, X)]$$

**Problem:** What if there are 100s or 1000s of potential confounders? Can we get $\sqrt{n}$-rates for the optimal price, even if $\beta_0$ is not $\sqrt{n}$ consistent?
We are given historical data of demand and price from a company.

**Goal:** Find a new optimal price policy based on the data.

**Approach:** Estimate demand function $Q(p, x)$ then optimize policy revenue

$$V(\pi) = E[\pi(X) \cdot Q(\pi(X), X)]$$

Can we get $\sqrt{n}$-rates for the optimal policy if policy space is simple, even if $\beta_0$ and $\alpha_0$ is not $\sqrt{n}$ consistent?
Policy Optimization from Observational Data with Continuous Actions

• Given observational data with \( n \) samples of triplets \((Y, T, X)\) of outcomes \( Y \), actions \( T \) and features \( X \)

• Assume conditional exogeneity: making an intervention and setting \( T = t, X = x \) yields value
\[
V(t, x) = E[Y|T = t, X = x]
\]

• Given a treatment policy space \( \Pi \): find a policy \( \hat{\pi} \) with good regret
\[
R(\Pi, n) = \sup_{\pi \in \Pi} E_X[V(\pi(X), X)] - E_X[V(\hat{\pi}(X), X)]
\]

Results extend if we optimize \( \rho(t, x) \cdot V(t, x) + c(t, x) \) for any known functions \( \rho, c \) or when \( Y \) is vector and optimize \( \rho(t, x)'V(t, x) + c(t, x) \)
Main Assumption

Linearity in known feature space:

\[ V(t, x) = \langle \theta_0(x), \phi(t, x) \rangle \]

for some known feature vector function \( \phi \) but unknown functions \( \theta_0 \)

**Example.** pricing; \( Y=\)demand, \( T=\)price

\[ V(t, x) = \theta_0(x) \cdot t + g_0(x) \]

and we optimize \( t \cdot V(t, x) \)

**Example.** investment allocation; \( Y=\)ROI, \( T=\)vector of investments

\[ V(t, x) = \langle \theta_0(x), t \rangle \]

and we optimize \( V(t, x) - c(t) \) for some known investment cost \( c \)
Mis-specification

Even if assumption is violated, we achieve regret wrt to best linear projection $V_p(t, x) = \langle \theta_p(x), \phi(t, x) \rangle$, where

$$\theta_p(x) = \arg\min_{\theta} E\left[ (V(t, x) - \langle \theta, \phi(t, x) \rangle)^2 \mid x \right]$$
Can we get $\sqrt{n}$-rates for the optimal policy if policy space is simple, even if $\theta_0$ is not $\sqrt{n}$ consistent?
Attempt 1: Direct Approach

• Estimate $\hat{\theta}$ by regressing $Y \sim T, X$ on one half of the data

• Optimize on the half part:

$$\hat{\pi} = \sup_{\pi \in \Pi} E_n[\langle \hat{\theta}(X), \phi(\pi(X), X) \rangle]$$

• Problem: estimate of policy value heavily depends on estimate of $\hat{\theta}$
• If estimate of $\hat{\theta}$ has RMSE of $\epsilon_n$, then regret incurs an error of $\epsilon_n$
Contribution 1: A Doubly Robust Method for Continuous Actions

• Estimate $\hat{\theta}$ by regressing $Y \sim T, X$ on one half of the data

• Estimate conditional covariance matrix on one half of the data

$$\hat{\Sigma}(x) = E[\phi(T, X)\phi(T, X)'] | X = x$$

• One the other half: construct a doubly robust estimate of the value coefficients:

$$\theta_{DR}(X) = \hat{\theta}(X) + \hat{\Sigma}^{-1}(X) \phi(T, X) (Y - \langle \hat{\theta}(X), \phi(T, X) \rangle)$$

• Optimize:

$$\hat{\pi} = \sup_{\pi \in \Pi} E_n[\langle \theta_{DR}(X), \phi(\pi(X), X) \rangle]$$
Double Robustness

\[ \theta_{DR}(X) = \hat{\theta}(X) + \hat{\Sigma}^{-1}(X) \phi(T, X) (Y - \langle \hat{\theta}(X), \phi(T, X) \rangle) \]

If \( \hat{\theta} \) is correct, then
\[ E[ \phi(T, X) (Y - \langle \hat{\theta}(X), \phi(T, X) \rangle) | X] = 0 \]
And
\[ E[\theta_{DR}(X) | X] = \hat{\theta}(X) = \theta_0(X) \]

If \( \hat{\Sigma} \) is correct, then
\[ \hat{\Sigma}^{-1}(X) \cdot E[\phi(T, X) \langle \hat{\theta}(X), \phi(T, X) \rangle | X] = \hat{\theta}(X) \]
and
\[ E[\theta_{DR}(X) | X] = \hat{\Sigma}^{-1}(X)E[\phi(T, X) E[Y | T, X] | X] = \Sigma_0^{-1}(X)E[\phi(T, X)\phi(T, X)' | X] \theta_0(X) = \theta_0(X) \]
Contribution 2: Semi-Parametric Efficiency Theorem. If we let
\[ \theta_{DR}^0(X) = \theta_0(X) + \Sigma_0^{-1}(X) \phi(T, X) (Y - \langle \theta_0(X), \phi(T, X) \rangle) \]
For any policy \( \pi \) the variance of the quantity off policy estimate
\[ E_n[\langle \theta_{DR}^0(X), \phi(\pi(X), X) \rangle] \]
is the best variance statistically achievable, without making further assumptions on the functions \( \theta_0 \); aka semi-parametric efficiency bound

* This holds either when the errors in the \( Y \) regression are homoscedastic, or when the model is mis-specified and \( \theta_0 \) is the best linear projection
Contribution 3a: Robust Regret

Theorem. If the RMSE of $\hat{\theta}$ and $\hat{\Sigma}^{-1}$ are $\epsilon_n$, then policy optimization based on the doubly robust estimate, achieves regret:

$$R(\Pi, n) = O(Rademacher(\Pi) + \epsilon_n^2)$$

If $Rademacher(\Pi) = O\left(\frac{1}{\sqrt{n}}\right)$ then as long as $\epsilon_n = o\left(n^{-1/4}\right)$ the impact from the estimates of $\theta$ and $\Sigma$ does not affect the leading regret term.
Contribution 3: Variance Based Robust Regret

Out-of-sample Regularized ERM:
Split your final sample in two.
Estimate optimal policy on first sample using the DR estimate
Consider the subset of the policy space that on the first sample has DR value at most the first solution plus some error $\mu_n$
Find the best policy within the subset based on the DR estimate on the second sample
Final Contribution: Variance Based Regret

Theorem. Consider the entropy integral

\[ \kappa(r, \Pi) \approx \int_0^r \sqrt{\frac{H_2(\epsilon, \Pi, n)}{n}} d\epsilon \]

Let \( V_2^0 \) denote the worst-case semi-parametric optimal variance of the difference between any two policies that are within \( \mu_n \) of the true optimal. Then the regret of out-of-sample regularized ERM is:

\[ R(\Pi, n) = O \left( \kappa \left( \sqrt{V_2^0}, \Pi \right) + \sqrt{\frac{V_2^0}{n}} + \epsilon_n^2 \right) \]

Example: for policies with constant VC dimension \( d \):

\[ R(\Pi, n) = O \left( \sqrt{\frac{V_2^0 d}{n}} + \epsilon_n^2 \right) \]
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$$V(\pi) = E[\pi(X) \cdot Q(\pi(X), X)]$$

Can we get $\sqrt{n}$-rates for the optimal policy if policy space is simple, even if $\beta_0$ and $\alpha_0$ is not $\sqrt{n}$ consistent?
Under homoscedastic observational policy

\[
\alpha_{DR}(X) = \hat{\alpha}(X) + \frac{T - E[T|X]}{\text{Var}(T - E[T|X])} \left( T - \hat{\alpha}(X)T - \hat{\beta}(X) \right)
\]

\[
\beta_{DR}(X) = \hat{\beta}(X) + \left( 1 + \frac{(T - E[T|X])E[T|X]}{\text{Var}(T - E[T|X])} \right) \left( T - \hat{\alpha}(X)T - \hat{\beta}(X) \right)
\]

So only need to regress \( T \sim X \) and estimate the variance of the residuals of this regression.
Sneak Peak of Experimental Results
Conclusions

• Addressed off policy optimization from observational data with continuous actions
• Under a linear of value assumption provided novel Doubly Robust off-policy estimate
• Showed semi-parametric efficiency of the variance of estimate
• Provided novel out-of-sample regularized ERM algorithm
• Showed variance-based regret with second order dependence from first stage regression and policy estimates