Crowdsourcing Contests with Private Types and Evaluation Uncertainty

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Abstract. We introduce a general model of crowdsourcing contests in which the performance of an agent is driven by his privately known ability type (adverse selection), his costly effort choice and it is further affected by evaluation uncertainty or luck (moral hazard). We show that when the marginal cost of effort is sufficiently large and the solvers' types, as well as, the evaluation uncertainty are independent and identically distributed respectively, there exists a unique Bayes Nash equilibrium effort. This equilibrium effort is in symmetric and pure strategies and involves inactive types. Our model includes as special cases the Tullock contest (Tullock, 1980), the all-pay auction model with private information of Moldovanu and Sela (2001), and the symmetric effort with evaluation uncertainty model of Lazear and Rosen (1981) that have been studied separately thus far. Our results suggest that several comparative statics results of the all-pay auctions with private information are robust in the presence of noisy feedback.

Key words: crowdsourcing, contests, existence, uniqueness, pure-strategy Bayes-Nash equilibrium, private information, moral hazard

1. Introduction

Despite the prevalence of crowdsourcing contests in practice the outcome of which is based on a privately known expertise level (ability) of each solver, his costly choice of effort and noisy feedback by the contest organizer (seeker), basic issues such as the existence and uniqueness of an equilibrium have been addressed only partially. Indeed, previous work focused on purely adverse selection models of the all-pay contest (see for example Moldovanu and Sela (2001) and the models discussed in Chapter 3 of Vojnović, 2016), or on purely moral hazard models (see Lazear and Rosen (1981) and Tullock (1980), and the models discussed in Chapter 4 of Vojnović, 2016). To our knowledge, little in general is known about the uniqueness of an equilibrium of a general crowdsourcing contest model that combines both adverse selection and moral hazard.

In this paper, we establish a condition that is sufficient for the uniqueness of the equilibrium of a general crowdsourcing contest model. Previous work by Chawla and Hartline (2013) has shown that the equilibrium of a purely adverse selection all-pay contest with independent and identically distributed (IID) valuations with linear effort cost is unique and involves pure, symmetric and strictly monotone strategies. The result of Chawla and Hartline is robust in the addition of noise affecting the rankings of the solvers: the unique equilibrium is shown to be in pure, symmetric and *weakly* monotone strategies. We consider a crowdsourcing contest model in which the cost of effort is a convex function and solvers' effort are perturbed by an IID noise with continuous and atomless density. In particular, equilibrium existence, but not uniqueness, is guaranteed to lie in weakly monotone pure strategies by the seminal work of Athey (2001). For our setting the result of Athey (2001) can be strengthened to show the existence of a *symmetric* equilibrium. As long as the marginal cost of effort is sufficiently large, we establish that the equilibrium is unique and may entail inactive types (Wärneryd, 2003; Levin and Smith, 1994; Wasser, 2013; Stouras, 2017). Generally, the equilibrium existence ensures that the model is consistent, while uniqueness strengthens the analysis of the optimal design of such a contest.

2. The model

A firm (*seeker*, "she") seeks to solve a problem through the use of a crowdsourcing contest. The seeker has a fixed budget r and elicits solutions from a population of n solvers who could potentially participate in the contest.

One of the benefits of crowdsourcing is that the seeker can tap into a pool of talent from outside solvers. However, these solvers exhibit significant heterogeneity in terms of their expertise. Each solver i is privately informed about his own *ability* (type) a_i. Abilities are independently drawn from a commonly known identical distribution $F(\cdot) \in C^1$ with an atomless density $f(\cdot)$ with support $[\underline{a}, \overline{a}]$, where $0 < \underline{a} < \overline{a} < \infty$.

Upon entry each solver i simultaneously chooses his *effort* $e_i(a_i) \geq 0$ at a cost[1] $\frac{c(e_i(a_i))}{a_i}$. In the special case where the cost of effort function is the identity, the solvers have a quasi-linear utility as it is standard in the mechanism design literature. We assume that $c(\cdot)$ is twice continuously differentiable with $c'(e) > 0$ and $c''(e) \geq 0$ for all $e > 0$, and with $c(0) = 0$ and $c'(0) > 0$. Further, we normalize solvers' outside option and entry fee to zero.

A solver with ability a_i who chooses effort $e_i(a_i)$ submits a solution with *performance* level X_i given by

$$X_i = e_i(a_i) + E_i, \qquad \text{(solvers' performance)}$$

where E_i is a random variable that models seeker's subjective taste, luck, or simply noise in observing solver i's effort perfectly. That is, the absolute performance of a solver is not known to him. We assume that the noise terms are identically and independently distributed from a common knowledge and atomless distribution $G(\cdot) \in C^2$ with zero mean, support $\mathbb{R}$ and density $g(\cdot)$. Further, assume that the limits $\lim_{x\to-\infty} g(x)$ and $\lim_{x\to+\infty} g(x)$ exist. An example of a distribution that satisfies these conditions is the Standard Normal distribution. Observe that when the variance of noise is zero, we obtain the observable effort model with private types

[1]This is without loss of generality since the ability type a_i is a known constant to solver i.

of Moldovanu and Sela (2001). Similarly, when the variance of solvers' ability is zero, solvers' performance is equivalent to the symmetric output model with moral hazard of Lazear and Rosen (1981).

To motivate solvers to participate and exert effort, the seeker allocates her budget r to the solver with the highest performance. That is, we *assume* that the winner of the contest is determined according to a *Winner-Takes-All* (WTA) allocation. That is, the *utility* of a solver i is either $r - \frac{c(e_i)}{a_i}$ if his performance ranks 1st out of n, or $-\frac{c(e_i)}{a_i}$ otherwise. Note that ties are events of measure zero. Using order statistics arguments, our analysis can be extended[2] to account for a weakly decreasing reward allocation in which $r_j \geq r_{j+1}$ for $j = 1, \ldots, N-1$.

In summary, the timing of the static game is as follows. The seeker announces her budget. Each solver privately learns his ability. The solvers simultaneously choose effort levels according to their ability types. Each submitted solution is affected by seeker's subjective taste. Finally, the solution with the highest performance is allocated the seeker's budget.

3. Analysis

Our static game has two main features. First, the effort choice of a solver is affected by his privately known ability, as well as, by the effort choices of the other solvers. Yet, each solver has only distributional information on the abilities of the other solvers. Second, a deterministic increase in effort of a solver is affected by noise which is outside the control of the solver. That is, given an ordered sequence of solvers' efforts, the addition of noise can entirely flip the rank-order of efforts with a positive probability.

The conventional approach in all-pay auctions to establishing the existence of a unique symmetric equilibrium effort entails three steps. First, a specific form of the equilibrium is conjectured. Second, the best response of an arbitrary solver i is found given the conjectured equilibrium strategy of the other solvers. Third, it is shown that the best response of solver i is unique and identical to the proposed equilibrium strategies of the rest solvers.

However, as we show below, this conventional approach has only a limited utility in our setting. The reasons are twofold. First, the co-existence of private information and moral hazard make the derivation of a symmetric equilibrium intractable in closed-form. Hence, it is problematic to establish the equilibrium uniqueness. Second, with the exception of very special cases, the second-order condition of the expected utility of a solver to be strictly negative at the proposed equilibrium may limit the region for which a unique equilibrium exists. It remains challenging to characterize properties of solvers' effort (such as monotonicity with respect to their type and comparative statics) without a closed-form of a proposed equilibrium.

[2] Our analysis remains valid for any *given* reward allocation chosen by the seeker. The related mechanism design question of the optimal reward allocation shall be attempted by the authors in future research.

As a result, we rely on general tools developed by Athey (2001) to establish the existence of an equilibrium in non-decreasing, pure and symmetric strategies. Then, we show that the sufficient conditions of Theorem 4 of Mason and Valentinyi (2010) are satisfied and hence the equilibrium is unique.

Theorem 1 (Equilibrium existence and uniqueness)**.** There exists a Bayes-Nash equilibrium effort in non-decreasing, pure and symmetric strategies. For a sufficiently large $c'(0)$ the equilibrium effort is unique.

The first part of Theorem 1 shows that a pure equilibrium effort exists under our framework §2 and it is weakly increasing in the ability types of the solvers. We prove that solvers' action space is bounded and that their expected utilities satisfy a specific single crossing property. Then, by the fixed point theorem of Athey (2001) our game of incomplete information with a continuum actions and continuous expected utilities must have an equilibrium in pure and non-decreasing strategies. In our setting, the result of Athey (2001) can be extended to establish the existence of a *symmetric* equilibrium in pure and non-decreasing strategies.

The second part of Theorem 1 establishes a sufficient condition for equilibrium uniqueness. In particular, we show that for a sufficiently large marginal cost of effort, each solver's marginal interim utility is strictly increasing in his ability type, and that each solver's type has larger impact on his marginal interim utility than the efforts of his opponents. This implies that given a solver's type, the best response correspondence is a contraction; hence, there is at most one equilibrium (Mason and Valentinyi, 2010). Taken together, the two parts of Theorem 1 imply the existence and uniqueness of an equilibrium effort in crowdsourcing contests with private types and evaluation uncertainty that satisfy our modeling assumptions (§2).

Importantly, Theorem 1 establishes a condition that rules out the existence of mixed or asymmetric equilibria in a broad class of noisy contests (all-pay auctions) with convex costs (payments). Chawla and Hartline (2013) present a structurally revealing argument for the non-existence of asymmetric equilibria in auction games where the rank-order of the bid of a solver is a deterministic function of his rank relatively to the rank of the bids of the other solvers. In our notation, Chawla and Hartline (2013) show that when the cost of effort function is the identity function, then asymmetric equilibria with pure continuous and strictly increasing strategies cannot be in Bayes-Nash equilibrium. We note that the proofs of Chawla and Hartline (2013) continue to hold in the absence of evaluation uncertainty when cost of effort is convex, or equivalently when solvers are ranked according to a concave function of effort. Further, in deterministic contests with IID types any mixed equilibrium is payoff equivalent to a pure equilibrium as it ensures equal probability to achieve a certain rank-order for almost all solver types. That is, the pure and symmetric structure of the equilibria of deterministic contests with IID types is robust in the addition of noisy feedback.

Strategies in Bayes-Nash equilibrium in deterministic contests are always strictly monotone in solvers' types (Chawla and Hartline, 2013). In contrast, if the performance rank-order of a solver is affected by a perturbation of his effort, the equilibrium effort is *weakly* monotone in general.

Proposition 1. *Assume that the sufficient condition for equilibrium uniqueness of Theorem 1 holds. There exists a unique solver with ability* $a_{min} \in [\underline{a}, \overline{a}]$ *such that* $e^*(a_i) = 0$ *for all* $a_i \in [\underline{a}, a_{min}]$*, and* $e^*(a_i) > 0$ *and strictly increasing for all* $a_i \in (a_{min}, \overline{a}]$*.*

Proposition 1 shows that noisy contests with a unique equilibrium induce screening of types. It is known that deterministic contests with strictly positive entry fees or outside options operate as a mechanism that restricts the entry of solvers of sufficiently low ability (see for example Moldovanu and Sela, 2001). Surprisingly, in noisy contests this holds even when the value of solvers' outside option is normalized to zero. The existence of screening of types implies that the solvers are a priori uncertain about the number of other solvers that would exceed the screening threshold a_{min} and will exert strictly positive effort (see for example Levin and Smith (1994), Moldovanu and Sela (2001) and Stouras et al. (2017)). That is, in noisy contests with a unique equilibrium, the number of participating solvers follows the Binomial distribution with population parameter n and entry probability $p = 1 - F(a_{min})$. We note that the presence of "inactive" types, or that the equilibrium effort in noisy contests is zero for a mass of types is also observed by Wärneryd (2003) in a common-value setting.

Our assumption of continuous types is a sufficient condition for solvers to play weakly monotone strategies in a symmetric equilibrium. When solvers types are discrete, we can recover this structural property when the noise has a sufficiently small support. To see that, suppose that there are $n = 2$ solvers with ability types either 0 (Low) with probability p_L or 1 (High) otherwise. Also, assume that the cost of effort is the linear function $c(e) = 3\,e$ and let $R = 1$. Then, in equilibrium $e_H^* \leq \frac{1}{3}$ and $e_L^* = 0$. Suppose that when a solver exerts effort, he is impacted by noise with support $(-0.1, 0.1)$. Since rankings in ability are preserved when ranked by costly effort and noise, the unique pure symmetric equilibrium is $e_L^* = e_H^* = 0$.

Next, we relate our general model (§2) to other well known models of the contest literature.

Corollary 1. *(i) Assume that solvers' ability is a constant almost surely, but solvers' effort is affected by noise. Then, solvers' expected utility is equivalent to the expected utility of Lazear and Rosen (1981) with a WTA reward allocation.*

(ii) Assume that the noise is zero almost surely for all solvers i, but solvers' ability is private information. Then, solvers expected utility is equivalent to the expected utility of Moldovanu and Sela (2001) with a WTA reward allocation.

A detailed proof is unnecessary. As the variance of ability $\mathbb{V}[A] \to 0$, the agent types become more homogeneous ex-ante and the equilibrium effort tends to the effort predicted by the symmetric model of Lazear and Rosen (1981). Similarly, as the variance of noise $\mathbb{V}[E_i] \to 0$, $a_{min} \to \underline{a}$ and rankings in ability are preserved by rankings in effort in equilibrium as in Moldovanu and Sela (2001).

We note that a winner-takes-all contest with ability homogeneous solvers and noise following an extreme value distribution is equivalent a standard Tullock contest. Luce and Suppes (1965) were the first to show that the extreme value distribution leads to the discrete choice model of logit. McFadden (1974) completed the equivalence by showing also the converse: the logit formula for the choice probability must imply that the unobserved utility is distributed according to the extreme value distribution.

To illustrate, consider contests with contest success function of Tullock (1980). For a discriminatory power $r > 0$, contest success function $\frac{e_i^r}{\sum_{j=1}^n e_j^r}$ and cost of effort $c(e) = e$ a Tullock contest can be represented as a contest with ability homogeneous solvers and noise from the Gumbel distribution. Following McFadden (1974) the probability of winning the contest is $\mathbb{P}\left[e_i \cdot E_i < e_{-i}^* \cdot E_{-i}\right]$ where $(E_i)_{i=1}^n$ are IID with CDF $F(\varepsilon) = \exp(-\varepsilon^{-r})$. By a simple transformation, this is equivalent to a contest with noise $\tilde{E}_i = \ln E_i$ following the Gumbel distribution $\tilde{F}(\tilde{\varepsilon}) = F(\exp(\tilde{\varepsilon}))$ and cost of effort $c(e) = \int_0^e x dx = \frac{e^2}{2}$. However, although solving the respective first-order conditions of these contests would lead to the same equilibrium effort, the standard Tullock contest and its associated noisy contest are *not* payoff equivalent. In particular, the equilibrium existence and uniqueness of the standard Tullock contest holds for a smaller range of discriminatory powers $r \leq \frac{n}{n-1}$, compared to its noisy contest counterpart.

4. Concluding remarks

While this paper has focused on the existence and uniqueness of the equilibrium effort of a crowdsourcing contest with private IID types and IID noise, our techniques can be used to establish the uniqueness of the equilibrium effort in contest models with purely private information or moral hazard. Future research to be attempted by the authors shall study whether noisy feedback benefits the objective of the contest seeker, as well as, the optimal design of such a noisy contest.

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Appendix A. Proofs

All proofs of the statements are given in the order of their appearance in the main text. Any additional results are stated and proved here.

Lemma 1. *Consider a continuous random variable with support* $\mathbb{R}$ *and a continuous probability density function* g *that is strictly positive on* $\mathbb{R}$*. Further, assume that the limits* $\lim_{x\to-\infty} g(x)$ *and* $\lim_{x\to+\infty} g(x)$ *exist. Then,* g *is uniformly bounded.*

Proof of Lemma 1. For any arbitrary $x \in \mathbb{R}$ fix an $\varepsilon > 0$ and consider the bounded interval $[x-\varepsilon,\, x+\varepsilon]$. By continuity of g there would exist a constant $k > 0$ such that $g(x) \le k$. Assume now that $\lim_{x\to+\infty} g(x)$ exists and that $\lim_{x\to+\infty} g(x) = +\infty$. This implies that as $x \to +\infty$ the corresponding cumulative distribution function $G(x) \to +\infty$, which is a contradiction. □

Proof of Theorem 1. *Existence.* Let $\overline{U}_i(a_i,\, e_i,\, e_{-i}^*) := \mathbb{E}\left[U_i(a_i,\, e_i,\, e_{-i}^*)\right]$. Note that $\overline{U}_i(a_i,\, e_i,\, e_{-i}^*) \le r - \frac{c(e_i)}{a_i}$ and that by choosing effort $e_i = 0$ solver i can guarantee himself a non-negative expected utility. Therefore, effort levels $e_i > c^{-1}(a_i \cdot r)$ are clearly dominated for solver i. That is, we can restrict the effort choice of solver i to the bounded interval $\left[0,\, c^{-1}(a_i \cdot r)\right]$.

We compare the performance of solver i against an arbitrary solver $j \ne i$. If each solver $j \ne i$ employs a strategy $e_j^*(A_j)$, the expected utility of solver i who has privately known ability a_i and exerts effort e_i is

$$\begin{aligned}
\mathbb{E}\left[U_i(a_i,\, e_i,\, e_{-i}^*)\right] &= r \cdot \mathbb{P}\left[X_i \text{ ranked 1st out of } n\right] - \frac{c(e_i)}{a_i} \\
&= r \cdot \prod_{j\ne i} \mathbb{P}\left[e_i + Z_i > e_j^*(A_j) + Z_j\right] - \frac{c(e_i)}{a_i} \\
&= r \cdot \prod_{j\ne i} \mathbb{P}\left[Z_j < e_i - e_j^*(A_j) + Z_i\right] - \frac{c(e_i)}{a_i} \\
&= r \cdot \int_{\mathbb{R}} \prod_{j\ne i} \mathbb{P}\left[Z_j < e_i - e_j^*(A_j) + x\right] dG(x) - \frac{c(e_i)}{a_i} \\
&= r \cdot \int_{\mathbb{R}} \left(\int_{\underline{a}}^{\overline{a}} \cdots \int_{\underline{a}}^{\overline{a}} G\left(e_i - e_j^*(a_j) + x\right) \prod_{j\ne i} dF(a_j) \right) dG(x) - \frac{c(e_i)}{a_i}
\end{aligned}$$

From the uniform bound $\frac{\partial^2 \overline{U}_i}{\partial e_i \partial a_i}(a_i,\, e_i,\, e_{-i}^*) = \frac{c'(e_i)}{a_i^2} > \frac{c'(0)}{\overline{a}^2} > 0$ the *Single Crossing Condition for games of incomplete information (SCC)* in Athey (2001) is satisfied. Our model is also consistent with assumption A1 in Athey (2001). Further, the action space of each solver is bounded and the expected utility $\overline{U}_i(a_i,\, e_i,\, e_{-i}^*)$ is continuous in $(e_i,\, e_{-i}^*)$ for all i. Hence, by Corollary 2.1 of Athey (2001) there exists an equilibrium in pure and non-decreasing strategies $e_i^*(a_i)$.

Next, we show existence of a *symmetric* equilibrium in pure and non-decreasing strategies. Theorem 1 in Appendix B of Kadan (2002) extends Corollary 2.1 of Athey (2001) and shows existence of an equilibrium in symmetric, pure and non-decreasing strategies for games of incomplete information when the action space is finite and types are independently drawn from the same distribution (IID). Then, Theorem 2 of Athey (2001) implies the existence of a symmetric equilibrium for games of incomplete information with a continuum of actions and IID types. Thus, we conclude the existence of an equilibrium in symmetric, pure and non-decreasing strategies for our setting.

Uniqueness. Due to independence, at a symmetric equilibrium the expected utility of solver i becomes

$$\mathbb{E}\left[U_i\left(a_i,\, e_i,\, e^*_{-i}\right)\right] = r \cdot \int_{\mathbb{R}} \left(\int_{\underline{a}}^{\overline{a}} G\left(e_i - e^*\left(a\right) + x\right)\, dF\left(a\right) \right)^{n-1} dG\left(x\right) - \frac{c\left(e_i\right)}{a_i}$$

We now provide a sufficient condition for the equilibrium to be unique. By our existence result above, this would guarantee that the unique equilibrium would be in symmetric, pure, and non-decreasing strategies. We focus on symmetric pure strategies and apply the Theorem 4 of Mason and Valentinyi (2010). We show that their assumptions (U1), (U2), (U3), (D1) and (D2) hold in our model.

(U1): By the Mean Value Theorem and the uniform bound $\frac{\partial^2 \overline{U}_i}{\partial e_i \partial a_i}\left(a_i,\, e_i,\, e^*_{-i}\right) > \frac{c'(0)}{\overline{a}^2} > 0$ we have that (U1) holds with $\delta := \frac{c'(0)}{\overline{a}^2}$ and the Euclidean metric $d_T\left(e_i,\, e_j\right) := \left|e_i - e_j\right|$.

(U2): We first show that $\frac{\partial \overline{U}_i}{\partial e_i}\left(a_i,\, e_i,\, e^*_{-i}\right)$ is uniformly bounded. Note that the first term on the RHS of

$$\frac{\partial \overline{U}_i}{\partial e_i}\left(a_i,\, e_i,\, e^*_{-i}\right) = r \cdot \int_{\mathbb{R}} \left(n-1\right) \left(\int_{\underline{a}}^{\overline{a}} G\left(e_i - e^*\left(a\right) + x\right)\, dF\left(a\right) \right)^{n-2} \cdot \left(\int_{\underline{a}}^{\overline{a}} g\left(e_i - e^*\left(a\right) + x\right)\, dF\left(a\right) \right) dG\left(x\right) - \frac{c'\left(e_i\right)}{a_i}$$

is positive and it is maximized at $e_i = c^{-1}\left(a_i \cdot r\right)$ and $e^*\left(a\right) = 0$ for all $a \in \left[\underline{a},\, \overline{a}\right]$. Set

$$L\left(a_i, e_i\right) := -\frac{c'\left(e_i\right)}{a_i}$$

and

$$U\left(a_i, e_i\right) := r \cdot \int_{\mathbb{R}} \left(n-1\right) \left(\int_{\underline{a}}^{\overline{a}} G\left(c^{-1}\left(\overline{a} \cdot r\right) + x\right)\, dF\left(a\right) \right)^{n-2} \cdot \left(\int_{\underline{a}}^{\overline{a}} g\left(c^{-1}\left(\overline{a} \cdot r\right) + x\right)\, dF\left(a\right) \right) dG\left(x\right) - \frac{c'\left(e_i\right)}{a_i}$$

We have that

$$L\left(a_i, e_i\right) \leq \frac{\partial \overline{U}_i}{\partial e_i}\left(a_i,\, e_i,\, e_{-i}\right) < U\left(a_i, e_i\right)$$

Then, by the Mean Value Theorem (U2) holds with

$$\omega := \max_{a_i,\, e_i} \left\{ L\left(a_i, e_i\right),\, U\left(a_i, e_i\right) \right\} > 0$$

(U3): By the Mean Value Theorem a sufficient condition for (U3) to hold is

$$\max_{e_i, e_{-i}} \frac{\partial \overline{U}_i}{\partial e_i}\left(a_i,\, e_i,\, e_{-i}\right) - \min_{e_i, e_{-i}} \frac{\partial \overline{U}_i}{\partial e_i}\left(a_i,\, e_i,\, e_{-i}\right) \le \kappa \quad \text{for all } i$$

That is, (U3) holds with

$$\kappa := r \cdot \int_{\mathbb{R}} (n-1) \left(\int_{\underline{a}}^{\overline{a}} G\left(c^{-1}\left(\overline{a} \cdot r\right) + x\right) dF\left(a\right) \right)^{n-2} \cdot \left(\int_{\underline{a}}^{\overline{a}} g\left(c^{-1}\left(\overline{a} \cdot r\right) + x\right) dF\left(a\right) \right) dG\left(x\right) > 0$$

(D1) As noted on p.28 in Mason and Valentinyi (2010) the assumption that solver types are independent implies that (D1) holds with $\iota := 0$.

(D2) By Lemma 1 g is uniformly bounded, i.e. there exists a constant $\nu > 0$ such that $g\left(x\right) \le \nu$ for all $x \in \mathbb{R}$. Then, (D2) holds for $\nu := \max_{x \in \mathbb{R}} g\left(x\right)$.

By Theorem 4 in Mason and Valentinyi (2010), if $\delta > \iota \cdot \omega + \nu \cdot \kappa$, or equivalently if

$$\frac{c'\left(0\right)}{\overline{a}^2} > \max_{x \in \mathbb{R}} g\left(x\right) \cdot r \cdot \int_{\mathbb{R}} (n-1) \left(\int_{\underline{a}}^{\overline{a}} G\left(c^{-1}\left(\overline{a} \cdot r\right) + x\right) dF\left(a\right) \right)^{n-2} \cdot \left(\int_{\underline{a}}^{\overline{a}} g\left(c^{-1}\left(\overline{a} \cdot r\right) + x\right) dF\left(a\right) \right) dG\left(x\right)$$

there exists a unique Bayes Nash equilibrium. □

Proof of Proposition 1. The first-order derivative $\frac{\partial \mathbb{E}\left[U_i\left(a_i, e_i, e^*\right)\right]}{\partial e_i}$ is equal to

$$r \cdot \int_{\mathbb{R}} (n-1) \left(\int_{\underline{a}}^{\overline{a}} G\left(e_i - e^*\left(a\right) + x\right) dF\left(a\right) \right)^{n-2} \left(\int_{\underline{a}}^{\overline{a}} g\left(e_i - e^*\left(a\right) + x\right) dF\left(a\right) \right) dG\left(x\right) - \frac{c'\left(e_i\right)}{a_i} \tag{1}$$

Assume that the sufficient condition for equilibrium uniqueness of Theorem 1 holds. Then, $\mathbb{E}\left[U_i\left(a_i,\, e_i,\, e^*\right)\right]$ is strictly concave in e_i. That is, (1) is strictly decreasing in e_i.

The first-order condition (FOC) of solver i's maximization problem is $\left.\frac{\partial \mathbb{E}\left[U_i\left(a_i, e_i, e^*\right)\right]}{\partial e_i}\right|_{e_i = e_i^*} \le 0$ with equality for ability a_i where $e^*\left(a_i\right) > 0$. Hence, if $e_i^* = e^*\left(a_i\right) > 0$, we must have that $e_j^* = e^*\left(a_j\right) > e^*\left(a_i\right)$ for all $a_j > a_i$ and $j \ne i$. Thus, a unique solver with ability $a_{min} \in \left[\underline{a},\, \overline{a}\right]$ must exist such that $e^*\left(a_i\right) = 0$ for all $a_i \in \left[\underline{a},\, a_{min}\right]$, and $e^*\left(a_i\right) > 0$ and strictly increasing for all $a_i \in \left(a_{min},\, \overline{a}\right]$. □