**6.853 Algorithmic Game Theory and Data Science** March 12, 2019

# Lecture 10

*Lecturer: Constantinos Daskalakis* *Scribe: Govind Ramnarayan*

## 1 Refresher

Last lecture we learned about mechanism design, in which we identify a desired outcome, then create a mechanism to achieve this outcome. We looked at games of incomplete information and some different types of equilibria that can occur in these games (e.g. dominant strategy, ex-post Nash, Bayesian Nash equilibria). We examined first-price and second-price (or Vickrey) auctions, and concluded that second-price auctions are optimal for maximizing social welfare.

Finally, we looked at the implementation of mechanisms. We saw what it meant for a function $f$ to be implemented in dominant strategies of a mechanism. We also distinguished direct versus indirect mechanisms, and saw that truthful, direct mechanisms are very powerful when it comes to implementing functions in dominant strategies (we called this the *Revelation Principle*).

## 2 Single-Dimensional Environments

In a single-dimensional environment, we have $n$ bidders. Each bidder $i$ has a private value $v_i$ (a *scalar*), which is her value for "being served." Finally, we have a feasible set $X$, where each element of $X$ is an $n$-dimensional 0/1 vector $(x_1, \dots, x_n)$, where $x_i$ denotes whether bidder $i$ is being served. Prominent examples of single-dimensional environments include $k$-unit auctions and sponsored search.

Recall that a mechanism is *direct* if the bidders have very simple action sets (i.e. reporting a scalar). We now define what we mean by a direct auction.

**Definition 1.** *A* direct auction *is defined by two rules:*

1. *An* allocation *rule* $x : \mathbb{R}^n \to \Delta(X)$.
2. *A payment rule* $p : \mathbb{R}^n \to \mathbb{R}^n$.

*Furthermore, the auction is executed as follows:*

1. *First, the bids* $b = (b_1, \dots, b_n)$ *are collected.*
2. *[**allocation**] Implement allocation* $x(b)$.
3. *[**payments**] Charge prices* $p(b)$.

Similar to last time, we define what it means to be DSIC.

**Definition 2** (DSIC (Dominant Strategy Incentive Compatible))**.** *A direct auction* $(x, p)$ *is* DSIC *iff for all* $i, b_{-i}$ *it is optimal for bidder* $i$ *to bids its true value. That is, for all* $z$ *and* $z'$*:*

$$z \cdot x_i(z, b_{-i}) - p_i(z, b_{-i}) \geq z \cdot x_i(z', b_{-i}) - p_i(z', b_{-i})$$

*where we abuse notation to define* $x_i(z, b_{-i})$ *as the probability under* $x(z, b_{-i})$ *that bidder* $i$ *is served.*

Finally, recall that the Revelation Principle (covered last time) tells us that any allocation rule that can be implemented in dominant strategies / ex-post Nash using an indirect mechanism can also be implemented using a direct, DSIC one.

## 2.1 Implementation in Single-Dimensional Environments

We now turn to discuss implementation in single-dimensional environments.

**Definition 3** (Implementable Allocation Rule)**.** *An allocation rule $x$ for a single-dimensional environment is* implementable *if there is a payment rule $p$ such that the sealed-bid auction $(x, p)$ is DSIC.*

In addition, we define what it means for an allocation rule to be *monotone*, which intuitively means that if a bidder $i$ bids more money, then they should be allocated more service (or at least, they should not be allocated less).

**Definition 4** (Monotone Allocation Rule)**.** *An allocation rule $x$ for a single-dimensional environment is* monotone *if for every bidder $i$ and bids $b_{-i}$ by the other bidders, the allocation $x_i(z, b_{-i})$ to $i$ is non-decreasing in $i$'s bid $z$.*

On the surface, implementable and monotone allocation rules may look completely unrelated. However, Myerson's Lemma tells us that they are actually the same!

**Lemma 1** (Myerson's Lemma)**.** *Fix a single-dimensional environment.*

1. *An allocation rule $x$ is implementable if and only if it is monotone.*
2. *If $x$ is implementable / monotone, there is an essentially* unique *payment rule such that the sealed-bid mechanism $(x, p)$ is DSIC, and this payment rule is given by the formula:*

$$\forall i, b_i : p_i(z; b_{-i}) = z \cdot x_i(z; b_{-i}) - \int_0^z x_i(t; b_{-i}) dt + p_i(0, b_{-i}) \tag{1}$$

3. *In particular, there is a unique payment function such that the mechanism is DSIC and additionally IR (individually rational) with* non-positive transfers *(that is, $b_i = 0$ implies that $p_i(b) = 0$, for any setting of $b_{-i}$).*

**Proof.** **(1.) Implementable $\Rightarrow$ monotone**: Recall that $x$ is implementable if there is a payment rule $p$ such that $(x, p)$ is DSIC. Hence, under the payment rule $p$, it makes sense for parties to bid their own value. Formally, for all $v_i, v_i'$, and $b_{-i}$, we have the following inequalities:

$$x_i(v_i, b_{-i}) \cdot v_i - p_i(v_i, b_{-i}) \geq x_i(v_i', b_{-i}) \cdot v_i - p_i(v_i', b_{-i}) \tag{2}$$

$$x_i(v_i', b_{-i}) \cdot v_i' - p_i(v_i', b_{-i}) \geq x_i(v_i, b_{-i}) \cdot v_i' - p_i(v_i, b_{-i}) \tag{3}$$

Together, inequalities 2 and 3 give us that

$$(x_i(v_i, b_{-i}) - x_i(v_i', b_{-i})) \cdot (v_i - v_i') \geq 0 \tag{4}$$

so the two terms on the left hand side are either both nonpositive or both nonnegative. Hence, for all $b_{-i}$, the function $x_i(\cdot, b_{-i})$ is non-decreasing.

**(2.) Implementable $\Rightarrow$ payment is essentially unique**: Fix $i$ and bids for the other parties $b_{-i}$. Since the auction is DSIC, we can specifically conclude that it is not worthwhile for the parties to bid slightly more (or less) than their true value. Let $u_i(v_i, b_{-i}) := x_i(v_i, b_{-i}) - p_i(v_i, b_{-i})$. Then it follows from DSIC that, for all $v$ and $\varepsilon > 0$:

$$u_i(v_i + \varepsilon, b_{-i}) \geq x_i(v_i, b_{-i}) \cdot (v_i + \varepsilon) - p_i(v_i, b_{-i}) \tag{5}$$

$$u_i(v_i, b_{-i}) \geq x_i(v_i + \varepsilon, b_{-i}) \cdot v_i - p_i(v_i + \varepsilon, b_{-i}) \tag{6}$$

Substituting in the definition of $u_i$ into inequalities 5 and 6 and rearranging, we get

$$u_i(v_i + \varepsilon, b_{-i}) - u_i(v_i, b_{-i}) \geq x_i(v_i, b_{-i}) \cdot \varepsilon \tag{7}$$

$$u_i(v_i + \varepsilon, b_{-i}) - u_i(v_i, b_{-i}) \leq x_i(v_i + \varepsilon, b_{-i}) \cdot \varepsilon \tag{8}$$

Combining inequalities 7 and 8 gives us that

$$x_i(v_i, b_{-i}) \cdot \varepsilon \le u_i(v_i + \varepsilon, b_{-i}) - u_i(v_i, b_{-i}) \le x_i(v_i + \varepsilon, b_{-i}) \cdot \varepsilon \tag{9}$$

If we take inequality 9, divide everything by $\varepsilon$, and use the definition of a derivative, we can see that it basically corresponds to $\frac{du_i}{dv_i} = x_i(v_i, b_{-i})$. Now we can finish the proof of this point.

$$\begin{aligned}
x \text{ implementable} &\Rightarrow x_i(\cdot, b_{-i}) \text{ non-decreasing} \\
&\Rightarrow x_i \text{ is Riemann integrable} \\
&\overset{(9)}{\Rightarrow} u_i(z, b_{-i}) - u_i(0, b_{-i}) = \int_0^z x_i(t, b_{-i})dt \\
&\Rightarrow p_i(z, b_{-i}) = x_i(z, b_{-i}) \cdot z - \int_0^z x_i(t, b_{-i})dt + p_i(0, b_{-i})
\end{aligned}$$

where the second-to-last line follows from the Fundamental Theorem of Calculus, and the last line follows from rearranging and substituting in the definition of $u_i$. Hence, the payment rule is essentially unique and given by the formula in the statement of Myerson's lemma.

**(3.) Implementable, NPT (non-positive transfers), and IR (individually rational) $\Rightarrow$ payment is unique.** Recall that NPT implies that $p_i(0, b_{-i}) \ge 0$ for all $b_{-i}$, since DSIC implies that the optimal strategy is to have $v_i = b_i$. Furthermore, IR implies that $p_i(0, b_{-i}) \le 0$ for all $b_i$. Together, these imply that $p_i(0, b_{-i}) = 0$ for all $b_i$, so the payment is unique.

**(1.) Monotone $\Rightarrow$ implementable** Suppose $x_i(\cdot, b_{-i})$ is non-decreasing for all $i$ and $b_{-i}$.

**Claim 1.** *Combined with payments as in (1), $(x, p)$ is DSIC.*

**Proof.** Fix $i, v_i$ (true type), $v_i'$ (candidate mis-report), and $b_{-i}$. Let

$$A = x_i(v_i, b_{-i}) \cdot v_i - p_i(v_i, b_{-i}) = \int_0^{v_i} x_i(t, b_{-i})dt$$

Now we show that the utility cannot rise when $i$ gives a candidate misreport $v_i'$. Indeed:

$$\begin{aligned}
B &= x_i(v_i', b_{-i}) \cdot v_i - p_i(v_i', b_{-i}) \\
&= x_i(v_i', b_{-i}) \cdot (v_i - v_i') + \int_0^{v_i'} x_i(t, b_{-i})dt \\
&= x_i(v_i', b_{-i}) \cdot (v_i - v_i') + \int_{v_i}^{v_i'} x_i(t, b_{-i})dt + \int_0^{v_i} x_i(t, b_{-i})dt \\
&\le x_i(v_i'.b_{-i}) \cdot (v_i - v_i') + (v_i' - v_i)x_i(v_i', b_{-i}) + \int_0^{v_i} x_i(t, b_{-i})dt \\
&\le A
\end{aligned}$$

where the second-to-last line follows from the fact that $x_i(\cdot, b_{-i})$ is non-decreasing, which lets us bound the integral above. So it is always worthwhile for $i$ to bid its true value. □

And combining all the points above completes the proof of Myerson's Lemma. □

We give an illustration of the reason why Myerson's payment rule yields a DSIC mechanism in Section 6.

Let us now give an interesting corollary of Myerson's Lemma.

**Corollary 1** (Corollary of Myerson's Lemma)**.** *The greedy allocation rule for sponsored search is implementable. Thus, there is a DSIC auction that maximizes social welfare (which we will see soon). However, in single-item settings, allocating to the second-highest bidder or the lowest bidder are both non-implementable.*

The final point of the corollary follows from the fact that allocating to the second-highest or lowest bidder both constitute non-monotone allocation rules. We will elaborate more about the sponsored search setting as we talk about applications of Myerson's Lemma.

# 3 Applications of Myerson's Lemma

## 3.1 Single Item Auction

Let us apply Myerson's Lemma to a single item auction. Due to monotonicity, we need to allocate the item to the highest bidder. Note that this means that the allocation rule must look as follows:

$$x_i(z, b_{-i}) = \begin{cases} 1 & \text{if } z > \max_{j \neq i} b_j \\ 0 & \text{otherwise} \end{cases}$$

Finally, applying the price rule given by Myerson's Lemma yields that we have

$$p_i(z, b_{-i}) = \begin{cases} \max_{j \neq i} & \text{if } z > \max_{j \neq i} b_j \\ 0 & \text{otherwise} \end{cases}$$

which is exactly the Vickrey auction!

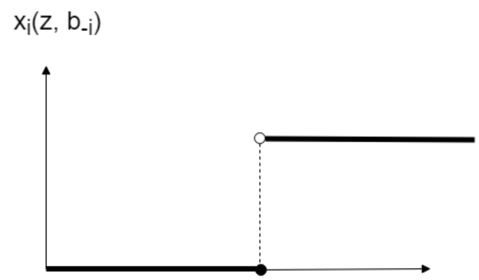


Figure 1: The allocation function given by Myerson's Lemma for a single-item auction.

## 3.2 Sponsored Search

We now describe how a sponsored search auction works. Just like the single-item auction, in a sponsored search auction we have $n$ bidders, who we will think of as advertisers. Unlike the single-item auction, we have $k$ slots for advertisements with (non-increasing) clickthrough rates $\alpha_1, \dots, \alpha_k$.

Our allocation rule will be to allocate the slots greedily based on the bidders' bids. Namely, the highest bidder should get the slot with $\alpha_1$, the next highest should get the slot with $\alpha_2$, etc. Using Myerson's Lemma, we get that the corresponding payment rule is

$$p_i(z, b_{-i}) = \sum_{\ell=1}^{k} (\alpha_\ell - \alpha_{\ell+1}) \cdot b_{-i}^{(\ell)} \cdot \chi_{z \geq b_{-i}^{(\ell)}}$$

where $b_{-i}^{\ell}$ denotes the $\ell^{th}$ highest bid among bidders $j \neq i$.

# 4 Single Item Revenue Maximization

Previously we looked at maximizing social welfare in single item auctions. Today we will look at the trickier question of revenue maximization.

First off, let us consider the following thought experiment. Suppose there is one bidder whose value for the item is $v$, and that is the highest value. Then if we were to give a take-it-or-leave-it offer, the

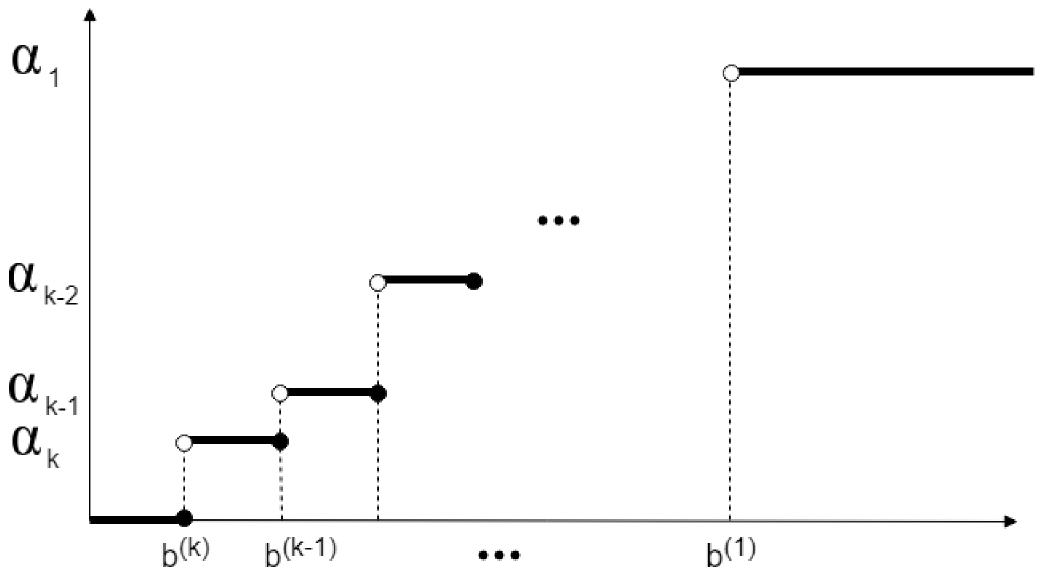


Figure 2: The allocation function given by Myerson's Lemma for a sponsored search auction, following the greedy algorithm. $b^{(m)}$ denotes the $m^{th}$ highest bid in $\{b_j, j \neq i\}$
, and the $\alpha_i$'s are the clickthrough rates for the respective slots.

highest achievable revenue is $v$. Unfortunately, the value $v$ is *private*, so how can we possibly price the item optimally?

This simple issue is actually quite fundamental. If we fix *any* price, there is some $v$ for which it fails to extract revenue $v$. This indicates that the bidder's private value is simply too strong of a benchmark for the auction to complete against.

We will solve this by assuming that the distribution $F$ over the bidder's value is known, and perform an average-case analysis over the distribution.

Let us consider a simple case with one bidder and one item. If we post price $r$, we get expected revenue $r \cdot (1 - F(r))$. So if $F$ is the uniform distribution on $[0, 1]$, the optimal choice of $r$ is $1/2$, which achieves expected revenue $1/4$. The optimal posted price is also called the *monopoly price*. Can we do better? Namely, in the same scenario with one bidder and a uniform value distribution over $[0, 1]$, is there any auction that does better than just posting the optimal price upfront?

Now suppose there are instead two bidders whose values are uniformly distributed over $[0, 1]$. The revenue of Vickrey's auction is the expectation of the minimum of the two uniform random variables, which gives us $1/3$. Is there anything we can do to do better?

Perhaps surprisingly, the answer is yes! By including a *reserve* price of $1/2$, we get expected revenue $5/12 > 1/3$[1]. Is $5/12$ the optimal expected revenue of *any* auction?

# 5 Revenue-Optimal Auctions

We will be interested in Bayesian, single-dimensional environments. In these, we have $n$ bidders, where each bidder $i$ has a private (scalar) value $v_i$, which is her value for being served, and $v_i$ is sampled from the distribution $F_i$, where $F_i$ is known to everyone (the auctioneer and other bidders). Finally, there is a feasible set $X$ of $n$-dimensional vectors $(x_1, \ldots, x_n)$ where $x_i$ denotes whether bidder $i$ is served. Examples of these include $k$-unit auctions and sponsored search.

When we are in this setting, it turns out that there are simple, direct, DSIC, and IR *revenue-optimal* auctions!

[1] Recall that a reserve price is a minimum price under which the seller refuses to sell the good.

**Theorem 1** (Myerson '81)**.** *Consider a single-dimensional setting, where the distribution $F_i$ of every bidder's private value is known to all other bidders as well as the auctioneer. Then there exists a* revenue-optimal *auction that is simple, direct, DSIC, and individually rational.*

Myerson's auction is *optimal*, in the sense that the expected revenue when all (which is in their best interest, since the auction is DSIC) is as large as the expected revenue of any other (potentially indirect) auction, when bidders use Bayesian Nash equilibrium strategies. In particular, the revenue is as large as that of any BIC direct mechanism.

**Theorem 2.** *Fix a Bayesian single-dimensional environment, where bidder distributions are $F_1, \ldots, F_n$ and $F = F_1 \times \ldots \times F_n$. Let $(x, p)$ be a BIC (Bayesian Incentive Compatible) mechanism satisfying interim IR and NPT. The expected revenue of this mechanism under truth-telling is*

$$\mathbb{E}_{v\sim F}[\sum_i p_i(v)] = \mathbb{E}_{v\sim F}[x_i(v)\varphi_i(v_i)] \tag{10}$$

*where $\varphi_i(v_i) := v_i - (1 - F_i(v_i))/f_i(v_i)$ is bidder $i$'s "virtual value function" (and $f_i$ denotes the density function for $F_i$).*

In particular, note that while the LHS of Equation 10 is the expected revenue, the RHS looks like an expected welfare, if the values are $\varphi_i(v_i)$. Hence, Equation 10 says that the expected revenue is equal to the expected virtual welfare, which we now prove.

**Proof.**

$$\begin{aligned}
\mathbb{E}_{v\sim F}[\sum_i p_i(v)] &= \sum_i \mathbb{E}_{v\sim F}[v_i \cdot x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(t, v_{-i})dt] \\
&= \sum_i \mathbb{E}_{v_i \sum F_i}[v_i \mathbb{E}_{v_{-i}}[x_i(v_i, v_{-i})] - \int_0^{v_i} \mathbb{E}_{v_{-i}}[x_i(t, v_{-i})]dt] \\
&= \sum_i \mathbb{E}_{v_i\sim F_i}[v_i \hat{x}_i(v_i) - \int_0^{v_i} \hat{x}_i(t)dt]
\end{aligned}$$

where $\hat{x}$ is the interim allocation to bidder $i$. Now we bring in our knowledge of the distribution of $v_i$:

$$\begin{aligned}
&= \sum_i \mathbb{E}_{v_i\sim F_i}[v_i \hat{x}_i(v_i)] - \sum_i \int_{v_i=0}^{+\infty} \int_{t=0}^{v_i} \hat{x}_i(t) f_i(v_i) dt dv_i \\
&= \sum_i \mathbb{E}_{v_i\sim F_i}[v_i \hat{x}_i(v_i)] - \sum_i \int_{t=0}^{+\infty} \int_{v_i=t}^{+\infty} \hat{x}_i(t) f_i(v_i) dt dv_i \\
&= \sum_i \mathbb{E}_{v_i\sim F_i}[v_i \hat{x}_i(v_i)] - \sum_i \int_{t=0}^{+\infty} \hat{x}_i(t)(1 - F_i(t))dt \\
&= \sum_i \int_{v_i=0}^{+\infty} v_i \cdot \hat{x}_i(v_i) f(v_i) dv_i - \sum_i \int_0^{+\infty} \hat{x}_i(v_i)(1 - F_i(v_i))dv_i \\
&= \sum_i \int_0^{+\infty} \hat{x}_i(v_i) \cdot \left(v_i - \frac{1 - F_i(v_i)}{f(v_i)}\right) f(v_i) dv_i \\
&= \sum_i \mathbb{E}_{v_i}[\hat{x}_i(v_i) \cdot \varphi_i(v_i)] \\
&= \mathbb{E}_{v\sim F}[\sum_i x_i(v) \cdot \varphi_i(v_i)]
\end{aligned}$$

□

# 6 Illustration of Monotone to Implementable

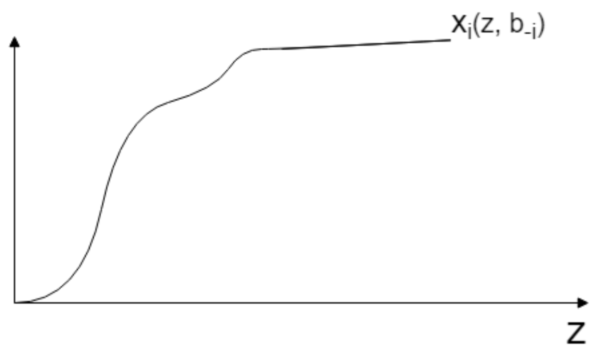


Figure 3: Example allocation function $x_i$ as a function of the value for bidder $i$.

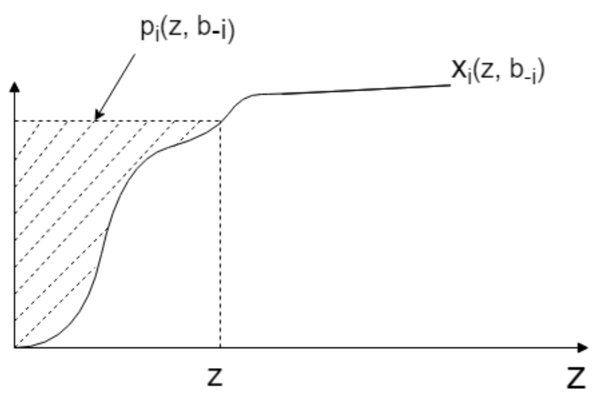


Figure 4: The shaded area is the payment for bidder $i$ under the payment rule given by Myerson's Lemma, given that their value is $z$.

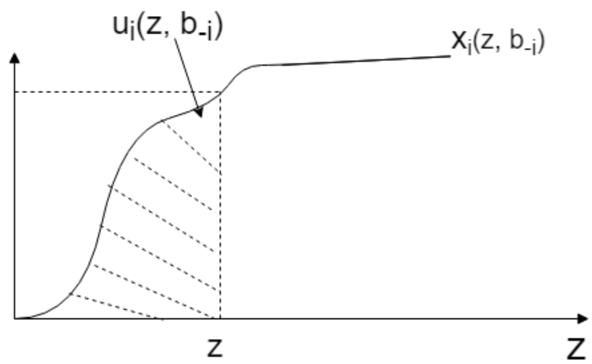


Figure 5: The shaded area is the *utility* for bidder $i$ under the payment rule given by Myerson's Lemma.

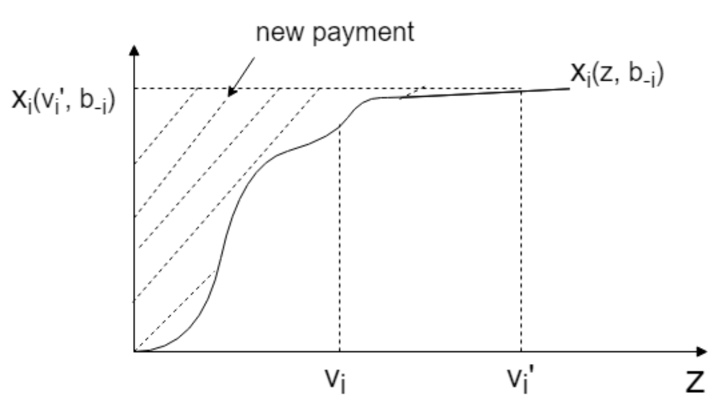


Figure 6: The shaded area is the *new payment* for bidder $i$ when they bid $v_i'$ instead of $v_i$.

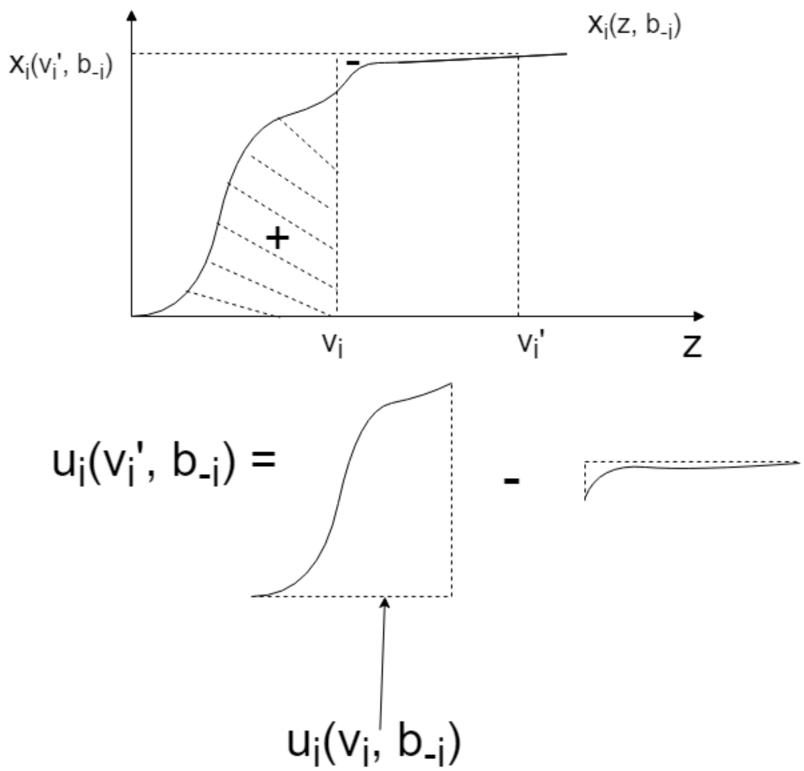


Figure 7: The shaded area represents the *new utility* for bidder $i$ when they bid $v_i'$ instead of their true value $v_i$. Notice that their old utility $u_i(v_i, b_{-i})$ is captured by the area labeled $+$, but then we subtract the area labeled -, which is contributed by the event that the price is higher than they truly value the item. Hence, we get that $u_i(v_i', b_{-i}) \leq u_i(v_i, b_{-i})$ under this payment rule, and bidders are incentivized to be truthful.