

Lecture 9

Lecturer: Constantinos Daskalakis

Scribe: Constantinos Daskalakis

The VCG (Vickrey-Clarke-Groves) Auction

1 Introduction

We have already seen Vickrey's auction for welfare optimization in single-item settings. We have also seen how to generalize it to single-dimensional environments, using that the greedy allocation rule in single-dimensional environments is monotone and hence implementable, and applying Myerson's payment identity. In this lecture, we provide a vast generalization of these mechanisms, applicable to general mechanism design environments with quasi-linear utilities, explained in the next section.

2 General Mechanism Design Setting

We consider a very general mechanism design setting with the following components:

- A set of *players/bidders/agents* $N := \{1, \dots, n\}$.
- A set of *outcomes/alternatives* A , which could be an arbitrary set.
- A *mechanism designer/auctioneer* whose goal is to choose some alternative.

Bidders and auctioneer are characterized by the following attributes:

- Every bidder has a private *valuation/type*, which is a function $v_i : A \rightarrow \mathbb{R}$, mapping alternatives to reals. Given $a \in A$, $v_i(a)$ represents bidder i 's value for alternative a . Each bidder's valuation v_i can be viewed as an element of \mathbb{R}^A .
- The tuple (v_1, \dots, v_n) is called the *type/valuation profile*. It is unknown to the auctioneer. Sometimes the auctioneer knows sets V_1, \dots, V_n with the guarantee that $v_i \in V_i$ for all i . Set V_i is called bidder i 's *typeset*. In a *Bayesian setting*, the auctioneer knows a distribution F_i over V_i , for all i . In this lecture, we consider non-Bayesian settings.
- The auctioneer is free to choose an *allocation rule* $a : (\mathbb{R}^A)^n \rightarrow A$ and a *payment rule* $p : (\mathbb{R}^A)^n \rightarrow \mathbb{R}^n$. The allocation rule maps type profiles to alternatives, while the price rule maps type profiles to a payment for each bidder. The tuple (a, p) is called a *direct mechanism*.
- Facing a direct mechanism, each bidder i reports some type b_i , which may or may not be his real type v_i . This is his *bid* or *report*. The tuple (b_1, \dots, b_n) is the *bid profile*.
- The utility of player i under bid profile \vec{b} is assumed to be $u_i(v_i, \vec{b}) = v_i(a(\vec{b})) - p_i(\vec{b})$. Such bidders, subtracting payment from value to compute their utility, are called *quasi-linear*.
- In this lecture we consider the goal of *welfare maximization*. In particular, the auctioneer's goal is to choose an alternative $a \in A$ such that $a \in \arg \max_{b \in A} \sum_{i \in N} v_i(b)$. The challenge is that the v_i 's are unknown to the auctioneer.

By the revelation principle, seen in previous lectures, we can restrict our attention to mechanisms where *truth-telling* is a dominant strategy equilibrium:

Definition 1. A direct mechanism (a, p) is dominant-strategy incentive-compatible (DSIC) / dominant-strategy truthful (DST) / strategy-proof if and only if $\forall i, \vec{b}_{-i}, v_i, b_i$ it holds that

$$v_i(a(v_i, \vec{b}_{-i})) - p_i(v_i, \vec{b}_{-i}) \geq v_i(a(b_i, \vec{b}_{-i})) - p_i(b_i, \vec{b}_{-i});$$

that is, truth-telling maximizes the bidder's utility, regardless of the other bidders' reports.

3 The VCG Mechanism

A clean approach towards welfare maximization is to fix the allocation rule to maximize welfare with respect to the reports:

$$a : \vec{b} \mapsto \arg \max_{o \in A} \sum_{i \in N} b_i(o)$$

and come up with a price rule that makes truth-telling a dominant strategy equilibrium. Indeed, if agents report their true valuations, and the allocation rule maximizes welfare with respect to the reports, then in fact the allocation rule is maximizing welfare with respect to the true types.

To make truth-telling a dominant strategy, we look at the way bidder i would optimize his bid, if he knew the other bids:

$$\text{bidder } i\text{'s maximization problem: } \vec{b}_{-i} \mapsto \arg \max_{b_i \in \mathbb{R}^A} v_i(a(b_i, \vec{b}_{-i})) - p_i(b_i, \vec{b}_{-i})$$

We are seeking a payment function such that, for all \vec{b}_{-i} :

$$v_i(a(v_i, \vec{b}_{-i})) - p_i(v_i, \vec{b}_{-i}) \geq \max_{b_i} \left\{ v_i(a(b_i, \vec{b}_{-i})) - p_i(b_i, \vec{b}_{-i}) \right\}.$$

Let us look for payments $p_i(\cdot)$ that only depend on b_i through the allocation, i.e. $p_i(\vec{b}) \equiv p_i(a(b_i, \vec{b}_{-i}))$. Then the above condition becomes:

$$v_i(a(v_i, \vec{b}_{-i})) - p_i(a(v_i, \vec{b}_{-i})) \geq \max_{b_i} \left\{ v_i(a(b_i, \vec{b}_{-i})) - p_i(a(b_i, \vec{b}_{-i})) \right\}.$$

Given our specialization of candidate payment rules to the form $p_i(a(b_i, \vec{b}_{-i}))$, the bidder can only affect his utility by manipulating the chosen allocation. He is thereby optimizing over allocations o in the image of $a(\cdot, \vec{b}_{-i})$ of the function: $v_i(o) - p_i(o)$. On the other hand, the auctioneer is optimizing over all allocations the function $b_i(o) + \sum_{j \neq i} b_j(o)$. To make truth telling a dominant strategy equilibrium, it suffices to choose a $p_i(\cdot)$ such that the two optimization problems are equivalent when $v_i = b_i$. That would guarantee that it's in the bidder's best interest to report $b_i = v_i$ since that would make the auctioneer optimize bidder i 's objective.

First Attempt

By setting the afore-described optimization problems equal when $b_i = v_i$, we get

$$p_i(\vec{b}) \equiv p_i(a(\vec{b})) = - \sum_{j \neq i} b_j(a(\vec{b})).$$

With these payments, the above discussion argues that it's in bidder i 's best interest to report $b_i = v_i$.

But we are not done yet. Even though we made truth-telling a dominant strategy, our mechanism may in fact pay the bidders to achieve this. Indeed, if all v_i 's are non-negative, our payment functions are always $p_i \leq 0$. For example, in a single-item setting, our payment rule charges the highest bidder 0, and pays all other bidders the highest bid. As making positive transfers to the bidders may be an unreasonable feature of our mechanisms, we seek ways to add more flexibility to the payment functions.

Second Attempt

To rectify the issue with our payments, we notice that adding any function of \vec{b}_{-i} to p_i does not affect the optimization problem that bidder i faces, and thereby does not affect the dominant strategy incentive compatibility of our mechanism.

So we can set $p_i(\vec{b}) = -\sum_{j \neq i} b_j(a(\vec{b})) + h_i(\vec{b}_{-i})$ for an arbitrary function h_i of \vec{b}_{-i} , without affecting that truth-telling is a dominant strategy equilibrium. We are now ready to define the VCG mechanism in its full generality.

Definition 2. A direct mechanism (a, p) is called a VCG mechanism if and only if the following conditions are satisfied:

- $\forall \vec{b}, a(\vec{b}) \in \arg \max_{o \in A} \sum_i b_i(o)$.
- \exists functions h_1, \dots, h_n where $h_i : (\mathbb{R}^A)^{n-1} \rightarrow \mathbb{R}$ such that $\forall \vec{b}, i, p_i(\vec{b}) = h_i(\vec{b}_{-i}) - \sum_{j \in N, j \neq i} b_j(a(\vec{b}))$.

We have already established the following:

Theorem 1. Any VCG mechanism is dominant-strategy truthful.

Recall that, if we do not specify the functions h_1, \dots, h_n properly, our mechanism may make positive transfers to the bidders. Moreover, it may not satisfy individual rationality at the truth-telling equilibrium, namely bidders may derive negative utility at the truth telling equilibrium. Are there functions h_1, \dots, h_n guaranteeing that a VCG mechanism makes no positive transfers, and satisfies individual rationality at the truth-telling equilibrium? The following conditions must be met:

$$p_i \geq 0 \Leftrightarrow h_i(\vec{b}_{-i}) \geq \sum_{j \in N, j \neq i} b_j(a(\vec{b}))$$

$$u_i \geq 0 \Leftrightarrow \sum_{j \in N} b_j(a(\vec{b})) \geq h_i(\vec{b}_{-i})$$

To satisfy the first inequality the simplest thing to do is to set $h_i(\vec{b}_{-i})$ equal to the maximum welfare of bidders in $N \setminus \{i\}$:

Definition 3. The Clarke payment rule requires that $h_i(\vec{b}_{-i}) = \max_{o \in A} \sum_{j \in N, j \neq i} b_j(o)$. So, in particular, the Clarke payment rule is: $p_i(\vec{b}) = \max_{o \in A} \sum_{j \in N, j \neq i} b_j(o) - \sum_{j \in N, j \neq i} b_j(a(\vec{b}))$.

Moreover, it is not hard to see that Clarke payments also satisfy individual rationality at truth-telling equilibrium, if the valuations are non-negative. We have shown the following:

Claim 1. Clarke payments satisfy $p_i(\vec{b}) \geq 0$, i.e. the non-positive transfers property (NPT). Additionally, if $\forall i, a \in A, v_i(a) \geq 0$, then $v_i(a(v_i, \vec{b}_{-i})) - p_i(v_i, \vec{b}_{-i}) \geq 0$, for all \vec{b}_{-i} .

We can now derive the VCG mechanism with the Clarke payment rule in the context of single-item auctions. We obtain the following mechanism:

- The item is allocated to some bidder $i^* \in \arg \max_i b_i$;
- The winner i^* pays the second highest bid $b^* = \max_{i \neq i^*} b_i$;
- All losers $i \neq i^*$ pay $v_{i^*} - v_{i^*} = 0$.

Observe that this is exactly the second-price auction! So, the second-price auction is a special case of the VCG mechanism with Clarke payments, in the single-item setting.

4 Procurement

In settings where bidder valuations may be negative, Clarke payments may lead to mechanisms that do not satisfy individual rationality at truth-telling equilibrium. We explore such settings in this section proposing mechanisms that rectify this issue.

Example 1 – Buying a path in a network. The goal is to set up an auction to buy a path of minimum cost from a node s to a node t in a network.

- $G = (V, E)$ with vertices s, t
- $A = \{\text{s-t paths}\}$
- Agents are the edges with costs c_e . In particular, if the chosen alternative a includes edge e , then the corresponding agent derives negative value: $v_e(a) = -c_e$; otherwise the agent's value is 0.
- **Goal:** Choose a path $a \in \arg \min_{a \in A} \sum_{e \in a} c_e \equiv \arg \max_a \sum_{e \in E} v_e(a)$

In this example the Clarke payment rule leads to a mechanism that does not satisfy individual rationality (see Remark 1). A DSIC, IR VCG auction for this procurement setting uses the following allocation and price rule:

$$a(c_{e_1}, \dots, c_{e_n}) : \text{choose the shortest } s\text{-}t \text{ path in } G \text{ according to edge-weights } c_e$$

(if there are several, then break ties in some fashion)

$$p_e(\vec{c}) = (-(\text{shortest path in graph } G \setminus e)) - (-(\text{length of chosen path not counting } e\text{'s cost}))$$

Notice that this payment function pays agents on the chosen path an amount at least as high as their cost, and pays 0 agents not on the chosen path.

Example 2 – Buying a Hamilton cycle in a network. The goal is to set up an auction to buy a cycle in a graph, which passes through every vertex exactly once and has minimum cost.

- $G = (V, E)$
- $A = \{\text{Hamilton cycles}\}$
- Agents are the edges with costs c_e . In particular, if the chosen alternative a includes edge e , then the corresponding agent derives negative value: $v_e(a) = -c_e$; otherwise the agent's value is 0.
- **Goal:** Choose a cycle $a \in \arg \min_a \sum_{e \in a} c_e \equiv \arg \max_a \sum_{e \in E} v_e(a)$

Again the Clarke payment rule leads to a mechanism that does not satisfy individual rationality. Mimicking our solution for Example 1, a DSIC, IR VCG auction for this procurement problem uses the following allocation and price rule:

$$a(c_{e_1}, \dots, c_{e_n}) : \text{choose the shortest Hamilton cycle according to edge weights } c_e \text{ (breaking ties in some fashion)}$$

$$p_e(\vec{c}) = (-(\text{shortest Hamilton cycle in } G \setminus e)) - (-(\text{length of chosen Hamilton cycle not counting } e\text{'s cost}))$$

Again, this payment function pays agents on the chosen cycle an amount at least as high as their cost, and pays 0 agents that are not on the chosen cycle.

Remark 1. A couple of remarks about Examples 1 and 2 are in order:

1. The payment functions proposed above look very similar to but are actually not Clarke payments. E.g. the Clarke payment for Example 1 would be

$$p_e^{\text{Clarke}}(\vec{c}) = (-(\text{shortest path in graph } G \text{ assuming } c_e = 0)) - (-(\text{length of chosen path not counting } e\text{'s cost}))$$

This payment rule pays agents on the chosen path 0, and may have agents outside of the chosen path pay. Hence it is not an IR mechanism. Indeed, notice that Claim 1 only guarantees IR when the valuations are non-negative, which in a procurement setting are clearly not. The payment functions used above rectify this issue in procurement settings.

2. *Although the above examples look very similar and in both cases $|A|$ is exponential in the size of G we see that in Example 1 both the allocation and the payment rule can be efficiently computed using a shortest path algorithm. On the other hand in Example 2 the allocation function requires solving an NP-complete problem, and therefore there unlikely exists an efficient algorithm to compute it. One idea to overcome this difficulty is to plug an approximation algorithm into the VCG mechanism. However, the VCG framework can actually not accommodate approximation algorithms, with the dominant strategy incentive compatibility failing when the required allocation problems cannot be solved exactly. One specific class of approximation algorithms that the VCG framework can directly accommodate are those that can be written as “maximal-in-range” algorithms: instead of maximizing over the whole set A these approximation algorithms maximize over a subset $A' \subset A$, over which the maximization problem can be solved efficiently and the optimal solution is guaranteed to be a good approximation to the optimal solution in the whole set A .*